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"The first attempt at a theory of transverse vibrations of elastic discs was made known by Sophie Germain. In 1811, she submitted a paper to the Paris Academy, which had announced a prize for such a theory. In her work, a hypothesis was made about the forces that resist the shape changes of a disc, and from this hypothesis, a partial differential equation for the vibrations was derived. The author made an error in her calculations; Lagrange, who was part of the committee evaluating the paper, derived the differential equation from her hypothesis, which had to give the correct results. This is the same equation that is still recognized as the correct one today. However, the boundary conditions were still missing, through which the solution of the partial differential equation would become specific. Sophie Germain derived these boundary conditions in a second paper, which she submitted to the Academy two years later, based on the same hypothesis. They were of such a nature that the author could find the solution of the problem for the case of rectangular discs. She compared her theoretical results for this case with observations and found a correspondence, which seemed to confirm her hypothesis. In a third paper, submitted to the Academy in 1815, she extended her hypothesis so that it could also derive the theory of vibrations of plates that are curved in their natural state. She was able to carry out the calculation for cylindrically curved plates and also found theoretical results here that were in agreement with experimental results."

"弹性圆盘横向振动理论的首次尝试由苏菲·热尔曼（Sophie Germain）提出。1811 年，她向巴黎科学院提交了一篇论文，该院为这种理论设立了奖项。在她的论文中，假设了抵抗圆盘形状变化的力，并从这个假设推导出了振动的偏微分方程。作者在计算中犯了一个错误；拉格朗日（Lagrange），当时参与评审该论文的委员会成员，从她的假设中推导出了偏微分方程，这个方程必须给出正确的结果。这正是现在仍被公认为正确的方程。然而，边界条件仍然缺失，这些条件使得偏微分方程的解变得具体化。苏菲·热尔曼在第二篇论文中，在相同的假设基础上推导出了这些边界条件，并于两年后将其提交给科学院。这些条件是这样的，作者能够为矩形圆盘的情况求出解。她将这一情况的理论结果与观察结果进行了比较，发现两者一致，这似乎证实了她的假设。在她于 1815 年提交给科学院的第三篇论文中，她扩展了她的假设，使得同样的理论可以推导出自然状态下弯曲的板材的振动理论。她能够对圆柱形弯曲的板材进行计算，并且也发现这些理论结果与实验结果相符。

The limits and the extent of the question of elastic surfaces, and the general equation of these surfaces" appeared in Paris in 1826.

Regardless of the confirmations that Sophie Germain's theory has experienced through experiments, it is not correct; because one can draw conclusions from it that are in obvious contradiction with reality. I limit myself to showing this by considering a plate that is in its natural state. The conclusions, which help Sophie Germain derive her laws for the deformation caused by external forces on such a plate and for the vibrations it undergoes, are essentially as follows.

In every element of the plate that has changed its shape, a force is generated that strives to return the same element to its original form. The condition of equilibrium of the plate is that the moment of all the forces generated in the same element and the moment of the given external forces yield a vanishing sum. Let ε be the thickness of the plate, df an element of its middle surface; the force generated in the element εdf will be greater, the greater the difference between the shape of df after the deformation and the original shape of this element. If one had a suitable measure for this difference, one could assume this force proportional to it; let u be such a measure, then this force would be
$N^2 u\, df$
where N^2 is a constant dependent on the thickness and the nature of the plate. The effort of this force goes towards reducing u; therefore, the moment of the same force would be:
$-N^2 u\, \delta u\, df;$
where δu denotes the virtual change of u.

If one applies the corresponding consideration to the case of an elastic rod, one arrives at the correct end equations when one sets $u =$ the reciprocal curvature radius of the middle line of the rod; Sophie Germain believed that in the case of a disk, $u =$ the sum of the reciprocal principal curvature radii of the middle surface could be assumed. If these principal curvature radii are φ_1 and φ_2, she obtained for the moment of the force generated in the element the expression
$-N^2 \left(\frac{1}{\varphi_1} + \frac{1}{\varphi_2}\right) \delta \left(\frac{1}{\varphi_1} + \frac{1}{\varphi_2}\right) df.$

“弹性表面问题的界限和范围，以及这些表面的一般方程”于1826年在巴黎出版。

尽管Sophie Germain的理论通过实验得到了证实，但它并不正确；因为从中可以得出与现实明显矛盾的结论。我仅限于通过考虑处于自然状态的板来展示这一点。通过这些结论，Sophie Germain推导出了外部力作用下板变形的定律以及板振动的定律，其本质如下。

在板的每一个形状发生变化的元素中，会产生一种力，这种力试图将该元素恢复到其原始形状。板的平衡条件是，在同一元素中产生的所有力的力矩和给定外力的力矩之和为零。设ε为板的厚度，df为其中间面的一个元素；在元素εdf中产生的力将随着变形后df的形状与该元素原始形状之间的差异增大而增大。如果有一个合适的度量来衡量这一差异，那么可以假设这种力与之成正比；设u为这样的度量，则该力为

$$N^2 u\, df$$

其中 N^2 是依赖于板的厚度和性质的常数。这种力的作用是减少 u；因此，该力的力矩为：

$$-N^2 u\, \delta u\, df;$$

其中 δu 表示 u 的虚拟变化。

如果将相应的考虑应用于弹性杆的情况，当设置 $u =$ 杆中间线的曲率半径的倒数时，可以得到正确的最终方程；Sophie Germain认为，在盘的情况下，$u =$ 中间面主曲率半径的倒数之和。如果这些主曲率半径为 φ_1 和 φ_2，她得到了生成力的力矩表达式

$$-N^2 \left(\frac{1}{\varphi_1} + \frac{1}{\varphi_2}\right) \delta \left(\frac{1}{\varphi_1} + \frac{1}{\varphi_2}\right) df.$$

And as the condition of equilibrium of the plate, the equation:

$$\delta P - N^2 \int \left(\frac{1}{\varphi_1} + \frac{1}{\varphi_2}\right) \delta \left(\frac{1}{\varphi_1} + \frac{1}{\varphi_2}\right) df = 0,$$

where δP denotes the moment of the given external forces.

To show that this condition cannot possibly be the correct one, I consider the case where a disk is brought very slightly out of its original shape by forces acting perpendicularly to its interior; for simplicity, I assume the edge of the disk to be free. The middle surface in its original shape is the xy plane of a rectangular coordinate system, z the displacement perpendicular to it, which the point (x, y) of the middle surface has suffered, Z the force acting in the direction of z on a line of the plate, which is pulled in the same direction through the point (x, y). Setting

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = u,$$

the equilibrium condition for u yields the partial differential equation

$$N^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = Z$$

and the boundary conditions

$$u = 0, \quad \frac{\partial u}{\partial n} = 0;$$

where n denotes the normal to the contour of the middle surface. Now the solution of the differential equation for u is already completely determined by the first of the two boundary conditions; it is therefore generally not possible to find an u that also satisfies the second boundary condition; and consequently there would be no equilibrium for the plate. If the given forces Z were such that a function u could be found that satisfies both boundary conditions, then one would have to determine the shape of the middle surface by substituting this value of u into the differential equation for z and determining z from the same. This equation is satisfied by infinitely many functions; it would therefore give infinitely many equilibrium states of the plate. This case would occur, for example, if no forces Z were present; if the plate were left in any shape, for which

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$

其中 n 表示中间面轮廓的法线。现在，根据第一个边界条件，微分方程的解已经完全确定；因此，一般情况下不可能找到同时满足第二个边界条件的 u；因此，板没有平衡状态。如果给定的力 Z 如此，以至于可以找到一个满足两个边界条件的函数 u，那么就需要通过将 u 的值代入 z 的微分方程来确定中间面的形状，并从同一方程中确定 z。这个方程被无限多个函数所满足；因此，它给出了板的无限多个平衡状态。例如，如果不存在力 Z，则会发生这种情况；如果板留在任何形状中，对于

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$

它将保持该形状不变。

并且作为板的平衡条件，方程为：

$$\delta P - N^2 \int \left(\frac{1}{\varphi_1} + \frac{1}{\varphi_2} \right) \delta \left(\frac{1}{\varphi_1} + \frac{1}{\varphi_2} \right) df = 0,$$

其中 δP 表示给定外力的力矩。

为了证明这个条件不可能是正确的，我考虑一个盘稍微偏离其原始形状的情况，通过垂直作用于其内部的力；为了简化起见，假设盘的边缘是自由的。中间面在其原始形状下是直角坐标系的 xy 平面，z 是点 (x, y) 的垂直位移，Z 是作用在板上的一条线上的力，该线在同一方向上被拉过点 (x, y)。设

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = u,$$

则 u 的平衡条件给出偏微分方程

$$N^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = Z$$

以及边界条件

$$u = 0, \quad \frac{\partial u}{\partial n} = 0;$$

without the intention of showing that it should return to its original shape. According to this equilibrium condition, the plate must be in equilibrium even if it has suffered finite curvatures, without the influence of external forces, as soon as for all points of its middle surface the sum of the reciprocals of the principal curvatures vanishes.

A second theory of the equilibrium and motion of elastic plates was proposed by Poisson and developed in his famous treatise "Sur l'équilibre et le mouvement des corps élastiques" [*]. However, this theory also requires a correction, which is my aim. Poisson arrived at his general equations of the equilibrium of elastic bodies by applying them to the case of a plate, leading to a partial differential equation similar to the hypothesis of Sophie Germain but with different boundary conditions, specifically three boundary conditions. I will prove that in general these cannot be simultaneously satisfied; from which it follows that according to the Poisson theory, a plate generally would have no equilibrium state. This proof will be given after I have derived the two boundary conditions that replace the three of Poisson's theory, because it naturally leads to the considerations through which I want to derive those conditions.

Poisson applied his theory to the case of a circular plate that vibrates so that all points equidistant from its center remain in the same state; he could apply it to this case because one of his three boundary conditions was identically fulfilled. From the modified theory, I will develop the laws of vibration of a free circular plate; in the special case mentioned, I will arrive at the same formulas that Poisson found. Through the kindness of Mr. Director Strehlke, who conducted measurements concerning the nodal lines of circular plates, I am able to compare some numerical results of the theory with the corresponding results of observation.

没有意图表明它应该恢复到其原始形状。根据这个平衡条件，即使板遭受了有限的曲率，在没有外力作用的情况下，只要中间表面所有点的主曲率倒数之和为零，板也必须处于平衡状态。

第二个关于弹性板平衡和运动的理论是由泊松提出的，并在他的著名论文“Sur l'équilibre et le mouvement des corps élastiques”[*]中发展起来的。然而，这个理论也需要修正，这是我的目的。泊松通过将他的弹性体平衡的一般方程应用于板的情况，得出了一个与索菲·热尔曼假设相似但具有不同边界条件的偏微分方程，具体来说是三个边界条件。我将证明在一般情况下这些条件不能同时满足；由此得出，根据泊松理论，板一般不会有平衡状态。这个证明将在导出替代泊松理论中三个边界条件的两个边界条件之后给出，因为这自然地引导到我想要从中推导这些条件的考虑。

泊松将其理论应用于一个圆形板振动的情况，使得所有离中心等距离的点保持在同一状态；他可以将其应用于这种情况，因为他的三个边界条件之一被完全满足。从修改后的理论中，我将推导自由圆形板振动的定律；在上述特殊情况下，我将得到与泊松相同的公式。通过斯特雷赫克先生（Director Strehlke）的好意，他进行了关于圆形板节点线的测量，我能够将理论的一些数值结果与相应的观测结果进行比较。

These equations apply to elastic bodies. These equations can be summarized in one, which states that the moment of the forces causing the deformation is equal to the variation of a certain integral. This equation also presupposes that it only applies when the displacements are infinitely small, while this exists as soon as the dilations and contractions are infinitely small; in the case of infinitely thin rods or plates that have suffered finite curvatures, these cannot be applied, but they can be. It is the following:

(1.) $0 = \delta P - \delta K \int dV(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)^2)$.

Here, δP denotes the moment of the given forces, dV the volume of an element of the body, λ_1, λ_2, λ_3 the principal dilations of this element; the integration extends over the entire body; K and θ are two constants, from which the elasticity coefficient q depends in such a way that
$q = 2K\frac{1+3\theta}{1+2\theta}$
is. One obtains the Poisson equations from (1.) by setting $\theta = \frac{1}{2}$, and the equations to which Mr. Wertheim was led by his experiments (*) by setting $\theta = 1$. If one denotes the rectangular coordinates of the point of dV in the original state of the body by x, y, z, the displacements in the directions of the axes that the same has suffered during the deformation by u, v, w, the forces acting on the same in the directions of the axes by XdV, YdV, ZdV; if one further denotes dO an element of the surface of the body and $(X)dO$, $(Y)dO$, $(Z)dO$ the pressure forces acting on the same in the directions of the axes, then the value of δP, which must be set in equation (1.), is as follows:

(2.) $\delta P = \int dV(X\partial u + Y\partial v + Z\partial w) + \int dO((X)\partial u + (Y)\partial v + (Z)\partial w))$;

where the first integral extends over the entire volume, the second over the entire surface of the body.

To be convinced that equation (1.) really provides the known equations for the deformations of elastic bodies in the case where u, v, w are infinitely small, one easily convinces oneself through the following calculation.

Let α, β, γ be the cosines of the angles which a line drawn through the point (x, y, z) with the coordinate axes forms: the dilation

这些方程适用于弹性体。这些方程可以总结为一个，它表明导致变形的力的力矩等于某个积分的变化。这个方程还假设它只在位移无限小的情况下适用，而这种情况一旦膨胀和收缩无限小时就存在；对于遭受有限曲率的无限薄杆或板，这些方程不能应用，但它们可以被应用。它是以下方程：

(1.) $0 = \delta P - \delta K \int dV(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)^2)$.

这里，δP 表示给定力的力矩，dV 是体元的体积，$\lambda_1, \lambda_2, \lambda_3$ 是该元素的主膨胀；积分在整个体上进行；K 和 θ 是两个常数，从这两个常数中弹性系数 q 依赖于
$q = 2K\frac{1+3\theta}{1+2\theta}$
的方式。从 (1.) 中设置 $\theta = \frac{1}{2}$ 可以得到泊松方程，通过设置 $\theta = 1$ 可以得到韦尔海姆先生通过实验得出的方程 (*)。如果用 x, y, z 表示体原始状态下的点 dV 的直角坐标，用 u, v, w 表示同一位置在变形过程中沿轴方向的位移，用 XdV, YdV, ZdV 表示在同一方向上作用的力；进一步设 dO 为体表面的一个元素，$(X)dO, (Y)dO, (Z)dO$ 为沿轴方向作用的压力力，则必须在方程 (1.) 中设置的 δP 的值如下：

(2.) $\delta P = \int dV(X\partial u + Y\partial v + Z\partial w) + \int dO((X)\partial u + (Y)\partial v + (Z)\partial w))$;

其中第一个积分在整个体积上进行，第二个积分在整个表面上进行。

为了确信当 u, v, w 无限小时，方程 (1.) 确实提供了已知的弹性体变形方程，人们很容易通过以下计算来说服自己。

设 α, β, γ 是通过点 (x, y, z) 并与坐标轴形成的角度的余弦：膨胀

The dilation in the direction of this line for the point (x, y, z), which is denoted by λ, is then determined under the assumption that u, v, w, and thus also the differential quotients of these quantities with respect to x, y, z, are infinitely small, by the following equation:

$$\lambda = \alpha^2 \frac{\partial u}{\partial x} + \beta^2 \frac{\partial v}{\partial y} + \gamma^2 \frac{\partial w}{\partial z}$$

$$+\beta\gamma \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) + \gamma\alpha \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + \alpha\beta \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right).$$

The principal dilations λ_1, λ_2, λ_3 are the values of λ for those values of α, β, γ for which the variation $\delta\lambda$ vanishes; i.e., they are the roots of the equation

$$0 = \left(\frac{\partial u}{\partial x} - \lambda \right) \left(\frac{\partial v}{\partial y} - \lambda \right) \left(\frac{\partial w}{\partial z} - \lambda \right)$$

$$-\tfrac{1}{4} \left(\frac{\partial u}{\partial x} - \lambda \right) \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 - \tfrac{1}{4} \left(\frac{\partial v}{\partial y} - \lambda \right) \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 - \tfrac{1}{4} \left(\frac{\partial w}{\partial z} - \lambda \right) \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2$$

$$+\tfrac{1}{4} \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right).$$

It follows from this:

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

and further

$$\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = \frac{\partial v}{\partial y}\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}$$

$$-\tfrac{1}{4}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 - \tfrac{1}{4}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \tfrac{1}{4}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2,$$

also

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2$$

$$+\tfrac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \tfrac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2.$$

These values of $\lambda_1 + \lambda_2 + \lambda_3$ and $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$ are to be substituted into equation (1.). We write it as follows:

$$(3.) \quad \delta P - K\delta\Omega = 0,$$

*) The equation resulting from this substitution is found in a slightly modified form in a paper by G. Green, "On the laws of reflexion and refraction of light," Camb. Phil. Trans. VII.; it is derived there in another way than here, without considering the principal dilations.

We set

$$(4.)\quad \Omega = \int dV(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)^2)$$

The variation $\delta\Omega$ will consist of three parts, the first depending on δu, the second on δv, and the third on δw; these three parts we denote by δR, δS, δT; then it follows that

$$(5.)\quad \delta\Omega = \delta R + \delta S + \delta T,$$

and one finds:

$$\delta R = \int dV \left\{2(1+\theta)\frac{\partial u}{\partial x} + 2\theta\frac{\partial v}{\partial y} + 2\theta\frac{\partial w}{\partial z}\right\}\frac{\partial \delta u}{\partial x}$$

$$+ \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial \delta u}{\partial y} + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\frac{\partial \delta u}{\partial z}\}.$$

From the expression for δR one obtains that of δS, and from this that of δT, when u, v, w and simultaneously x, y, z are cyclically permuted. The expression for δR is decomposed into three integrals, of which the first has under the integral sign the factor $\frac{\partial \delta u}{\partial x}$, the second the factor $\frac{\partial \delta u}{\partial y}$, the third the factor $\frac{\partial \delta u}{\partial z}$. For the first of the three integrals, apply the theorem expressed by the equation

$$\int dV F\frac{\partial G}{\partial x} = -\int dV G\frac{\partial F}{\partial x} - \int dO F G\cos(N, x),$$

where F and G denote two arbitrary functions of x, y, z, dV the element of a bounded space, dO the element of the surface of the same, and (N, x) the angle formed by the normal to the interior of the bounded space with the x-axis; for the second and third integrals, whose sum is δR, apply the theorems that arise from the above through permutation of x with y or z; in each case let $G = \delta u$. If we now combine the integrals after dV and those after dO, it follows that

$$\delta R = -\int dV \left\{ 2(1+\theta)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + (1+2\theta)\frac{\partial^2 v}{\partial x \partial y} + (1+2\theta)\frac{\partial^2 w}{\partial x \partial z} \right\} \delta u$$

$$-\int dO \left\{ 2(1+\theta)\frac{\partial u}{\partial x} + 2\theta\frac{\partial v}{\partial y} + 2\theta\frac{\partial w}{\partial z} \right\} \cos(N, x)$$

$$+\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\cos(N, y) + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\cos(N, z)\}\delta u.$$

我们设

$$(4.) \quad \Omega = \int dV(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)^2)$$

变化 $\delta\Omega$ 将由三部分组成，第一部分依赖于 δu，第二部分依赖于 δv，第三部分依赖于 δw；这三部分我们分别记为 δR 、δS、δT；于是得到

$$(5.) \quad \delta\Omega = \delta R + \delta S + \delta T,$$

并且找到：

$$\delta R = \int dV \left\{ 2(1+\theta)\frac{\partial u}{\partial x} + 2\theta\frac{\partial v}{\partial y} + 2\theta\frac{\partial w}{\partial z} \right\} \frac{\partial \delta u}{\partial x}$$

$$+\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\frac{\partial \delta u}{\partial y} + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\frac{\partial \delta u}{\partial z}\}.$$

从 δR 的表达式中可以得到 δS 的表达式，进而得到 δT 的表达式，当 u, v, w 和同时 x, y, z 循环置换时。 δR 的表达式分解为三个积分，其中第一个积分符号下的因子是 $\frac{\partial \delta u}{\partial x}$，第二个是 $\frac{\partial \delta u}{\partial y}$，第三个是 $\frac{\partial \delta u}{\partial z}$。对于这三个积分中的第一个，应用以下方程表示的定理

$$\int dV F\frac{\partial G}{\partial x} = -\int dV G\frac{\partial F}{\partial x} - \int dO F G \cos(N, x),$$

其中 F 和 G 表示 x, y, z 的任意函数，dV 是有限空间的元素，dO 是同一表面的元素，(N, x) 是有限空间内部法线与 x 轴形成的角；对于第二和第三个积分，其和为 δR，应用通过 x 与 y 或 z 置换而产生的定理；每次令 $G = \delta u$。如果我们现在组合 dV 后的积分和 dO 后的积分，那么得到

$$\delta R = -\int dV \left\{2(1+\theta)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + (1+2\theta)\frac{\partial^2 v}{\partial x \partial y} + (1+2\theta)\frac{\partial^2 w}{\partial x \partial z}\right\}\delta u$$

$$-\int dO \left\{2(1+\theta)\frac{\partial u}{\partial x} + 2\theta\frac{\partial v}{\partial y} + 2\theta\frac{\partial w}{\partial z}\right\}\cos(N, x)$$

$$+\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\cos(N, y) + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\cos(N, z)\}\delta u.$$

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The expressions for δS and δT, and then the equation (5.) determine the value of $\delta\Omega$; set this, as well as the value of δP from (2.), in the equation (3.), which combines the integrals that extend over the volume of the body, as well as those that refer to its surface, and set, according to the principles of the calculus of variations, the factors of δu, δv, δw under the two integral signs $= 0$; thus one obtains the following equations:

For a point inside the body:

$$K\left\{2(1+\theta)\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + (1+2\theta)\frac{\partial^2 v}{\partial x \partial y} + (1+2\theta)\frac{\partial^2 w}{\partial x \partial z}\right\} + X = 0,$$
$$K\left\{2(1+\theta)\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 v}{\partial x^2} + (1+2\theta)\frac{\partial^2 w}{\partial y \partial z} + (1+2\theta)\frac{\partial^2 u}{\partial y \partial x}\right\} + Y = 0,$$
$$K\left\{2(1+\theta)\frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + (1+2\theta)\frac{\partial^2 u}{\partial z \partial x} + (1+2\theta)\frac{\partial^2 v}{\partial z \partial y}\right\} + Z = 0.$$

And for a point on the surface:

$$K\left\{2(1+\theta)\frac{\partial u}{\partial x} + 2\theta\frac{\partial v}{\partial y} + 2\theta\frac{\partial w}{\partial z}\right\}\cos(N, x) + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\cos(N, y) + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\cos(N, z) + (X) = 0,$$
$$K\left\{2(1+\theta)\frac{\partial v}{\partial y} + 2\theta\frac{\partial w}{\partial z} + 2\theta\frac{\partial u}{\partial x}\right\}\cos(N, y) + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\cos(N, z) + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos(N, x) + (Y) = 0,$$
$$K\left\{2(1+\theta)\frac{\partial w}{\partial z} + 2\theta\frac{\partial u}{\partial x} + 2\theta\frac{\partial v}{\partial y}\right\}\cos(N, z) + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\cos(N, x) + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\cos(N, y) + (Z) = 0.$$

These equations are the same as those derived by Cauchy in a way where he did not rely on the consideration of molecular forces; they transform into the Poisson's equations when $\theta = \frac{1}{2}$, and into the Wertheim's equations when $\theta = 1$.

I will now give a derivation of equation (1.), from which it will follow that it has a more general validity than the equations (6.). Considerations similar to those that follow here have been made.

表达式 δS 和 δT，以及方程 (5.) 确定了 $\delta\Omega$ 的值；将此值以及从 (2.) 中的 δP 值代入方程 (3.)，该方程结合了扩展到体内的积分，以及与表面相关的积分，并根据变分法的原则，在两个积分符号下设置 δu、δv、δw 的因子为 0；从而得到以下方程：

对于体内的一个点：

$$K\left\{2(1+\theta)\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+(1+2\theta)\frac{\partial^2 v}{\partial x\partial y}+(1+2\theta)\frac{\partial^2 w}{\partial x\partial z}\right\}+X=0,$$
$$K\left\{2(1+\theta)\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial x^2}+(1+2\theta)\frac{\partial^2 w}{\partial y\partial z}+(1+2\theta)\frac{\partial^2 u}{\partial y\partial x}\right\}+Y=0,$$
$$K\left\{2(1+\theta)\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+(1+2\theta)\frac{\partial^2 u}{\partial z\partial x}+(1+2\theta)\frac{\partial^2 v}{\partial z\partial y}\right\}+Z=0.$$

对于表面上的一个点：

$$K\left\{2(1+\theta)\frac{\partial u}{\partial x}+2\theta\frac{\partial v}{\partial y}+2\theta\frac{\partial w}{\partial z}\right\}\cos(N,x)+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\cos(N,y)+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\cos(N,z)+(X)=0,$$
$$K\left\{2(1+\theta)\frac{\partial v}{\partial y}+2\theta\frac{\partial w}{\partial z}+2\theta\frac{\partial u}{\partial x}\right\}\cos(N,y)+\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\cos(N,z)+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\cos(N,x)+(Y)=0,$$
$$K\left\{2(1+\theta)\frac{\partial w}{\partial z}+2\theta\frac{\partial u}{\partial x}+2\theta\frac{\partial v}{\partial y}\right\}\cos(N,z)+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\cos(N,x)+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\cos(N,y)+(Z)=0.$$

这些方程与 Cauchy 通过不依赖于分子力考虑的方式得出的方程相同；当 $\theta=\frac{1}{2}$ 时，它们转化为 Poisson 方程，当 $\theta=1$ 时，转化为 Wertheim 方程。

我现在将给出方程 (1.) 的推导，从中可以得出它具有比方程 (6.) 更广泛的适用性。类似于此处后续的考虑已被提出。

Lagrange has repeatedly used this in his mechanics, for example, in the derivation of the equilibrium condition of an elastic rod.

Let dV be the volume of an infinitely small part of the elastic body in its natural state. The state in which this part is through deformation can be known from the natural as follows: that the part without changing the relative position of its molecules has another position in space and then dilated in three mutually perpendicular directions (uniformly in each, but differently in the different ones). An infinitely small sphere becomes an ellipsoid whose axes are the directions in which the dilations took place. These dilations are therefore the principal dilations λ_1, λ_2, λ_3. The elasticity of the body causes the considered part to strive to contract in the directions in which it is extended; the forces with which it strives to contract are $L_1 dV$, $L_2 dV$, $L_3 dV$; the first of them seeks λ_1, the second λ_2, the third λ_3 to reduce. The moment of the first force is therefore $-L_1 dV \delta\lambda_1$, that of the second $-L_2 dV \delta\lambda_2$, that of the third $-L_3 dV \delta\lambda_3$, and the total moment of the three forces is
$$-dV(L_1\delta\lambda_1 + L_2\delta\lambda_2 + L_3\delta\lambda_3).$$

Now L_1, L_2, L_3 are functions of λ_1, λ_2, λ_3. We know from them that they simultaneously disappear with λ_1, λ_2, λ_3; further, that L_1 is a symmetric function of λ_2 and λ_3, and the same function of λ_1, λ_2, λ_3 as L_2 of λ_2, λ_3, λ_1 and L_3 of λ_3, λ_1, λ_2 must be. If one therefore takes λ_1, λ_2, λ_3 as infinitely small, one obtains
$$L_1 = a\lambda_1 + b\lambda_2 + b\lambda_3,$$
$$L_2 = b\lambda_1 + a\lambda_2 + b\lambda_3,$$
$$L_3 = b\lambda_1 + b\lambda_2 + a\lambda_3$$
where a and b denote two quantities dependent on the nature of the body. If one introduces instead of these two other K and θ, which are connected with them by the equations
$$a = 2K(1+\theta), \quad b = 2K\theta,$$
one obtains for the moment of the forces generated in dV the expression
$$-dV \cdot \delta K(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)^2).$$

*) I will designate a compression as negative dilation.

拉格朗日在他的力学中多次使用了这种方法，例如，在推导弹性杆的平衡条件时。

设 dV 是弹性体在其自然状态下的无限小部分的体积。这个部分通过变形达到的状态可以从自然状态如下得知：该部分在不改变其分子相对位置的情况下，在空间中有另一个位置，然后在三个相互垂直的方向上被拉伸（每个方向均匀拉伸，但在不同的方向上不同）。一个无限小的球体变成一个椭球体，其轴是发生拉伸的方向。这些拉伸因此是主拉伸 λ_1，λ_2，λ_3。身体的弹性使所考虑的部分在它被拉伸的方向上努力收缩；它努力收缩的力是 L_1dV，L_2dV，L_3dV；第一个力寻求 λ_1，第二个力寻求 λ_2，第三个力寻求 λ_3 减少。第一个力的力矩因此为 $-L_1dV\delta\lambda_1$，第二个力的力矩为 $-L_2dV\delta\lambda_2$，第三个力的力矩为 $-L_3dV\delta\lambda_3$，这三个力的总力矩为
$-dV(L_1\delta\lambda_1 + L_2\delta\lambda_2 + L_3\delta\lambda_3)$.

现在 L_1，L_2，L_3 是 λ_1，λ_2，λ_3 的函数。我们从它们知道它们与 λ_1，λ_2，λ_3 同时消失；进一步地，L_1 是 λ_2 和 λ_3 的对称函数，并且是 λ_1，λ_2，λ_3 的相同函数，就像 L_2 是 λ_2，λ_3，λ_1 的函数，以及 L_3 是 λ_3，λ_1，λ_2 的函数一样。因此如果取 λ_1，λ_2，λ_3 为无限小，得到
$L_1 = a\lambda_1 + b\lambda_2 + b\lambda_3,$
$L_2 = b\lambda_1 + a\lambda_2 + b\lambda_3,$
$L_3 = b\lambda_1 + b\lambda_2 + a\lambda_3$
其中 a 和 b 表示依赖于身体性质的两个量。如果引入另外两个 K 和 θ，它们由以下方程连接
$a = 2K(1+\theta), \quad b = 2K\theta,$
则在 dV 中生成的力的力矩表达式为
$-dV \cdot \delta K(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)^2).$

*) 我将压缩称为负拉伸。

And for the moment of all forces generated in the body, the expression is:
$-\delta K \int dV(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)^2)$.

The sum of this moment and the moment of the external forces must vanish for the equilibrium state; as expressed by equation (1).

Section 2

Now we want to consider a plate. We assume that its base surfaces in their natural state are formed by two parallel, infinitely close planes, and its edge by an arbitrary cylindrical surface that cuts these perpendicularly. The plate has undergone a shape change due to forces acting on its interior and pressure forces exerted on its edge, while its base surfaces remain free. These forces are ultimately only so large that the dilations they produce can be considered infinitesimally small. However, it is not specified that the curvatures the plate has experienced are infinitesimally small; these we will provisionally consider finite.

To apply equation (1) to the case of such a plate, we make two assumptions which we regard as results of experiments and which fully correspond to those made by Jacob Bernoulli regarding an elastic rod; namely the following:

> Every straight line of the plate which was originally perpendicular to the base surfaces remains straight and perpendicular to the surfaces which were originally parallel to the base surfaces during the shape change;
>
> All elements of the middle surface (i.e., the plane which in the natural state of the plate is the plane parallel to the base surfaces and located in the middle between them) undergo no dilation during the shape change.

With the help of these two assumptions, the values of the principal dilations λ_1, λ_2, λ_3 for the current case can be easily expressed through the principal radii of curvature of the middle surface. For this purpose, let us consider a further examination of an elastic body of arbitrary form. Imagine in this body in its original state an infinitely small sphere with a diameter a and a diameter plane A perpendicular to a; during the shape change, the sphere transforms into an ellipsoid; the molecules lying on a and on A then lie on

对于在体内生成的所有力的力矩，表达式为：
$-\delta K \int dV(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \theta(\lambda_1 + \lambda_2 + \lambda_3)^2)$.

这个力矩与外力的力矩之和在平衡状态下必须消失；如方程 (1) 所示。

第二节

现在我们转向对板的考虑。假设其基面在自然状态下由两个平行的无限接近的平面形成，其边缘由任意圆柱表面形成，该表面垂直切割这些平面。板通过作用在其内部的力和施加在其边缘的压力力而发生形变，同时其基面自由。这些力最终只是如此之大，以至于它们产生的拉伸可以被视为无限小。然而，并没有说明板所经历的曲率是无限小的；这些曲率我们暂时视为有限的。

为了将方程 (1) 应用于这种板的情况，我们做出两个假设，这些假设被视为实验结果，并且完全符合 Jacob Bernoulli 关于弹性杆的假设；具体如下：

> 板上每一条原本垂直于基面的直线，在形变后保持直线并垂直于那些原本平行于基面的平面；
>
> 中间层的所有元素（即在自然状态下板的中间层，位于两基面之间的中点）在形变过程中不经历任何拉伸。

借助这两个假设，主拉伸 λ_1，λ_2，λ_3 的值对于当前情况可以通过中间层的主曲率半径轻松表示。为此目的，考虑一个具有任意形状的弹性体。设想在这个体的原始状态下有一个无限小的球体，其直径为 a，并且有一个垂直于 a 的直径平面 A；在形变过程中，球体变为椭球体；位于 a 和 A 上的分子随后位于

with a diameter a' and on a diametral plane A' of the ellipsoid, which are conjugate to each other. Generally, therefore, a' will not be perpendicular to A'. If a' is perpendicular to A', then a' will be equal to one of the principal axes of the ellipsoid, and the maximum and minimum of the two other perpendicular diameters will be equal to the two other principal axes; i.e., the dilation of a' will be one of the principal dilations, and the two other principal dilations will be the maximum and minimum of the two other perpendicular dilations.

If we apply this theorem to the case of the plate, it follows from assumption (1): that for any point inside the plate, the dilation in the direction of the normal drawn through it to the middle surface is one of the principal dilations. Let us call z the original distance of the considered point from the middle surface, z' its distance after the shape change; simultaneously let z' also denote the normal of the middle surface after the shape change with respect to its position. Set

$$z' - z = q,$$

so $\frac{\partial q}{\partial z}$ will be the value of a principal dilation. Since this should be infinitely small, and since q simultaneously vanishes with z, q must be infinitely small against z.

Considering assumption (2.), it is seen that the dilation in any direction perpendicular to z' is $-\frac{z'}{\varrho}$, when ϱ is the radius of curvature of the curve in which the plane through z' and the relevant direction cuts the middle surface at the footpoint of z'. If we call the radii of curvature of the principal sections of the middle surface for the footpoint of z', ϱ_1 and ϱ_2, then $\frac{z'}{\varrho_1}$ and $\frac{z'}{\varrho_2}$ are the values of the two other principal dilations. Since $z' - z$ is infinitely small against z, we can write for these also $\frac{z}{\varrho_1}$ and $\frac{z}{\varrho_2}$.

The values of the principal dilations λ_1, λ_2, λ_3, which have now been found, substitute into equation (1.). Expressing the element of the volume of the plate by $df \cdot dz$, where df is the element of the middle surface, we obtain the equation

$$0 = \delta P - K\delta \iint dfdz \left[\left(\frac{\partial q}{\partial z}\right)^2 + \left(\frac{z}{\varrho_1}\right)^2 + \left(\frac{z}{\varrho_2}\right)^2 + \theta \left(\frac{\partial q}{\partial z} + \frac{z}{\varrho_1} + \frac{z}{\varrho_2}\right)^2 \right].$$

具有直径 a' 和椭球体的直径平面 A'，这两个是共轭的。一般来说，因此 a' 不会垂直于 A'。如果 a' 垂直于 A'，那么 a' 将等于椭球体的一个主轴，而两个其他垂直直径的最大值和最小值将等于另外两个主轴；即 a' 的膨胀将是其中一个主膨胀，而另外两个主膨胀将是两个其他垂直膨胀的最大值和最小值。

如果我们把这个定理应用到板的情况，从假设 (1) 可以得出：对于板内任意一点，在通过该点向中间表面绘制的法线方向上的膨胀是一个主膨胀。我们称 z 为所考虑点与中间表面的原始距离，z' 为其形状变化后的距离；同时让 z' 也表示形状变化后中间表面的法线相对于其位置。设

$$z' - z = q,$$

所以 $\frac{\partial q}{\partial z}$ 将是主膨胀的值。由于这应该是无限小的，并且 q 同时随着 z 消失，因此 q 必须是对 z 无限小的。

考虑到假设 (2.)，可以看出在任何垂直于 z' 的方向上的膨胀是 $-\frac{z'}{\varrho}$，当 ϱ 是通过 z' 和相关方向切割中间表面曲线的曲率半径时。如果我们将中间表面在 z' 脚点的主截面的曲率半径称为 ϱ_1 和 ϱ_2，则 $\frac{z'}{\varrho_1}$ 和 $\frac{z'}{\varrho_2}$ 是另外两个主膨胀的值。由于 $z' - z$ 对 z 是无限小的，我们也可以写成 $\frac{z}{\varrho_1}$ 和 $\frac{z}{\varrho_2}$。

现在找到的主膨胀值 λ_1，λ_2，λ_3，代入方程 (1.) 中。用 $df \cdot dz$ 表示板体积的元素，其中 df 是中间表面的元素，我们得到方程

$$0 = \delta P - K\delta \iint dfdz \left[\left(\frac{\partial q}{\partial z}\right)^2 + \left(\frac{z}{\varrho_1}\right)^2 + \left(\frac{z}{\varrho_2}\right)^2 + \theta \left(\frac{\partial q}{\partial z} + \frac{z}{\varrho_1} + \frac{z}{\varrho_2}\right)^2 \right].$$

If the thickness of the plate is denoted by 2ε, then the integration with respect to z should be taken from $z = -\varepsilon$ to $z = \varepsilon$.

Now we want to show that one of the principal dilations $\frac{\partial q}{\partial z}$ can be expressed through the two others $\frac{z}{\varrho_1}$ and $\frac{z}{\varrho_2}$ without needing to know the forces which caused the shape change of the plate. To this end, we need to examine more closely the value of δP, which is given by equation (2.). The right-handed coordinate system, on which this equation is based, should be chosen such that the xy plane is the middle surface in the natural state of the plate; then z retains the meaning given to it here. Let us denote the angles which the normal of the middle surface z' forms with the coordinate axes by (z', x), (z', y), (z', z), the original coordinates of the footpoint of z' by x_0, y_0, 0, and the displacements which this point has suffered in the directions of the axes by u_0, v_0, w_0; according to assumption (1.), we have:

$x + u = x_0 + u_0 + z' \cos(z', x)$,
$y + v = y_0 + v_0 + z' \cos(z', y)$,
$z + w = w_0 + z' \cos(z', z)$,

or, since $z' - z$ is infinitely small against z:

$x + u = x_0 + u_0 + z \cos(z', x)$,
$y + v = y_0 + v_0 + z \cos(z', y)$,
$z + w = w_0 + z \cos(z', z)$.

From this, it follows:

$$\begin{cases} \delta u = \delta u_0 + z\delta \cos(z', x), \\ \delta v = \delta v_0 + z\delta \cos(z', y), \\ \delta w = \delta w_0 + z\delta \cos(z', z). \end{cases}$$

These values of δu, δv, δw are to be substituted into equation (2.). The second parts of these are infinitely small compared to the first parts because they contain the factor z; we have retained them for the case of the above considerations not to exclude the integrals

$\int_{-\varepsilon}^{+\varepsilon} X z\, dz, \int_{-\varepsilon}^{+\varepsilon} Y z\, dz, \int_{-\varepsilon}^{+\varepsilon} Z z\, dz, \int_{-\varepsilon}^{+\varepsilon} (X) z\, dz, \int_{-\varepsilon}^{+\varepsilon} (Y) z\, dz, \int_{-\varepsilon}^{+\varepsilon} (Z) z\, dz$

of the same order as the integrals

$\int_{-\varepsilon}^{+\varepsilon} X\, dz, \int_{-\varepsilon}^{+\varepsilon} Y\, dz, \int_{-\varepsilon}^{+\varepsilon} Z\, dz, \int_{-\varepsilon}^{+\varepsilon} (X)\, dz, \int_{-\varepsilon}^{+\varepsilon} (Y)\, dz, \int_{-\varepsilon}^{+\varepsilon} (Z)\, dz.$

如果用 2ε 表示板的厚度，则关于 z 的积分应从 $z=-\varepsilon$ 到 $z=\varepsilon$ 进行。

现在我们想要证明，其中一个主膨胀 $\frac{\partial q}{\partial z}$ 可以通过另外两个 $\frac{z}{\varrho_1}$ 和 $\frac{z}{\varrho_2}$ 来表达，而无需知道导致板形状变化的力。为此，我们需要更仔细地研究由方程 (2.) 给出的 δP 的值。该方程所基于的右手坐标系应选择为：在板的自然状态下，xy 平面是中间表面；则 z 保留其在此处赋予的意义。设中间表面 z' 与坐标轴形成的角分别记为 (z',x)，(z',y)，(z',z)，脚点 z' 的原始坐标为 x_0，y_0，0，且该点在轴方向上的位移分别为 u_0，v_0，w_0；根据假设 (1.)，我们有：

$$x+u=x_0+u_0+z'\cos(z',x),$$
$$y+v=y_0+v_0+z'\cos(z',y),$$
$$z+w=w_0+z'\cos(z',z),$$

或，由于 $z'-z$ 对 z 是无限小的：

$$x+u=x_0+u_0+z\cos(z',x),$$
$$y+v=y_0+v_0+z\cos(z',y),$$
$$z+w=w_0+z\cos(z',z).$$

由此得出：

$$\begin{cases}\delta u=\delta u_0+z\delta\cos(z',x),\\ \delta v=\delta v_0+z\delta\cos(z',y),\\ \delta w=\delta w_0+z\delta\cos(z',z).\end{cases}$$

这些 δu，δv，δw 的值应代入方程 (2.) 中。这些值的第二部分相对于第一部分是无限小的，因为它们包含因子 z；我们保留它们是为了不排除以下积分：

$$\int_{-\varepsilon}^{+\varepsilon} Xz\,dz, \int_{-\varepsilon}^{+\varepsilon} Yz\,dz, \int_{-\varepsilon}^{+\varepsilon} Zz\,dz, \int_{-\varepsilon}^{+\varepsilon}(X)z\,dz, \int_{-\varepsilon}^{+\varepsilon}(Y)z\,dz, \int_{-\varepsilon}^{+\varepsilon}(Z)z\,dz$$

与以下积分同阶：

$$\int_{-\varepsilon}^{+\varepsilon} X\,dz, \int_{-\varepsilon}^{+\varepsilon} Y\,dz, \int_{-\varepsilon}^{+\varepsilon} Z\,dz, \int_{-\varepsilon}^{+\varepsilon}(X)\,dz, \int_{-\varepsilon}^{+\varepsilon}(Y)\,dz, \int_{-\varepsilon}^{+\varepsilon}(Z)\,dz.$$

If the values of δu, δv, δw from (8.) are substituted into (2.), it is seen that δP is independent of δq; it follows that the term dependent on δq in the second part of equation (7.) must vanish for itself. This leads to:

$$(1+\theta)\frac{\partial q}{\partial z}+\theta\left(\frac{z}{\varrho_1}+\frac{z}{\varrho_2}\right)=0$$

or

$$\frac{\partial q}{\partial z}=-\frac{\theta}{1+\theta}\left(\frac{z}{\varrho_1}+\frac{z}{\varrho_2}\right).$$

Substituting this value of $\frac{\partial q}{\partial z}$ into equation (7.), we get:

$$0=\delta P-K\delta\iint dfdzz^2\left\{\frac{1}{\varrho_1^2}+\frac{1}{\varrho_2^2}+\frac{\theta}{1+\theta}\left(\frac{1}{\varrho_1}+\frac{1}{\varrho_2}\right)^2\right\},$$

or, when the integration with respect to z is carried out:

$$(9.)\quad 0=\delta P-\frac{2}{3}\varepsilon^3K\delta\iint df\left(\frac{1}{\varrho_1^2}+\frac{1}{\varrho_2^2}+\frac{\theta}{1+\theta}\left(\frac{1}{\varrho_1}+\frac{1}{\varrho_2}\right)^2\right).$$

§ 3.

The general equilibrium condition for a plate derived above will now be applied to the case treated by Poisson: namely, the case where the plate has only moved infinitesimally away from its original equilibrium position. We begin with the further development of the value of δP.

Let w_0 be an infinitely small quantity of first order; then, since the middle surface has not suffered any dilations, u_0 and v_0 must be infinitely small quantities of second order; hence δu_0 and δv_0 can be considered as infinitely small compared to δw_0. The equations (8.) can thus be written as:

$$\delta u = z\delta\cos(z', x),$$

$$\delta v = z\delta\cos(z', y),$$

$$\delta w = \delta w_0 + z\delta\cos(z', z).$$

Furthermore, in the present case, if only infinitely small quantities of first order are considered:

$$\cos(z', x) = -\frac{\partial w_0}{\partial x_0}, \quad \cos(z', y) = -\frac{\partial w_0}{\partial y_0}, \quad \cos(z', z) = 1;$$

and therefore:

$$\delta u = -z\frac{\partial \delta w_0}{\partial x_0}, \quad \delta v = -z\frac{\partial \delta w_0}{\partial y_0}, \quad \delta w = \delta w_0.$$

These values are substituted into equation (2.). If we again express:

如果将 $\delta u, \delta v, \delta w$ 的值从 (8.) 代入 (2.)，可以看出 δP 与 δq 无关；由此得出，在方程 (7.) 的第二部分中依赖于 δq 的项必须自行消失。这导致：

$$(1+\theta)\frac{\partial q}{\partial z}+\theta\left(\frac{z}{\varrho_1}+\frac{z}{\varrho_2}\right)=0$$

或

$$\frac{\partial q}{\partial z}=-\frac{\theta}{1+\theta}\left(\frac{z}{\varrho_1}+\frac{z}{\varrho_2}\right).$$

将此值代入方程 (7.) 中，我们得到：

$$0=\delta P-K\delta\iint dfdzz^2\left\{\frac{1}{\varrho_1^2}+\frac{1}{\varrho_2^2}+\frac{\theta}{1+\theta}\left(\frac{1}{\varrho_1}+\frac{1}{\varrho_2}\right)^2\right\},$$

或，当对 z 进行积分时：

$$(9.)\quad 0=\delta P-\frac{2}{3}\varepsilon^3 K\delta\iint df\left(\frac{1}{\varrho_1^2}+\frac{1}{\varrho_2^2}+\frac{\theta}{1+\theta}\left(\frac{1}{\varrho_1}+\frac{1}{\varrho_2}\right)^2\right).$$

上述导出的板的一般平衡条件现在将应用于泊松处理的情况：即板仅从其原始平衡位置移动了无穷小的情况。我们从 δP 的进一步发展开始。

设 w_0 是一阶无穷小量；那么，由于中间表面没有遭受任何膨胀，u_0 和 v_0 必须是二阶无穷小量；因此 δu_0 和 δv_0 可以认为相对于 δw_0 是无穷小的。方程 (8.) 可以写为：

$$\delta u = z\delta\cos(z', x),$$

$$\delta v = z\delta\cos(z', y),$$

$$\delta w = \delta w_0 + z\delta\cos(z', z).$$

此外，在当前情况下，如果只考虑一阶无穷小量：

$$\cos(z', x) = -\frac{\partial w_0}{\partial x_0}, \quad \cos(z', y) = -\frac{\partial w_0}{\partial y_0}, \quad \cos(z', z) = 1;$$

因此：

$$\delta u = -z\frac{\partial \delta w_0}{\partial x_0}, \quad \delta v = -z\frac{\partial \delta w_0}{\partial y_0}, \quad \delta w = \delta w_0.$$

这些值被代入方程 (2.) 中。如果我们再次表达：

Express the volume element of the plate by $dfdz$ and the surface element of its edge by $dsdz$, where ds is the contour element of its middle surface, and for convenience write w, x, y instead of w_0, x_0, y_0. Thus we obtain:

$$\delta P = \iint dfdz \left\{ Z\delta w - z \left(X\frac{\partial \delta w}{\partial x} + Y\frac{\partial \delta w}{\partial y} \right) \right\}$$

$$+ \iint dsdz \left\{ (Z)\delta w - z \left((X)\frac{\partial \delta w}{\partial x} + (Y)\frac{\partial \delta w}{\partial y} \right) \right\}.$$

The forces X, Y, Z and the pressure forces (X), (Y), (Z) can be considered here as independent of w because w is infinitely small. We want to transform the expression for δP. It can initially be written in the following way:

$$\delta P = \iiint dxdydz \left\{ Z + z \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \right\} \delta w$$

$$- \iiint zdxdydz \frac{\partial X \delta w}{\partial x} - \iiint zdxdydz \frac{\partial Y \delta w}{\partial y}$$

$$+ \iint dsdz \left\{ (Z)\delta w - z \left((X)\frac{\partial \delta w}{\partial x} + (Y)\frac{\partial \delta w}{\partial y} \right) \right\}.$$

The second and third of these four integrals can be transformed by applying the formulas:

$$\left\{\begin{array}{l}\iint dxdy\frac{\partial F}{\partial x}=-\oint ds\cos\varphi F,\\ \iint dxdy\frac{\partial F}{\partial y}=-\oint ds\sin\varphi F.\end{array}\right.$$

Here F denotes an arbitrary function of x and y; the double integration is over a bounded surface, the single one over the contour of the same; φ denotes the angle that the normal directed towards the interior of the bounded surface makes with the positive x-axis when it is rotated in the direction in which the normal is described until it becomes parallel to the positive x-axis (until it must be rotated so that after a rotation of 90° it takes the position of the positive y-axis). By using these formulas, one obtains for δP the following value:

$$\delta P=\iiint dxdydz\left\{Z+z\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)\right\}\delta w$$

$$+\iint dsdz\left\{(Z)+z\left((X)\cos\varphi+(Y)\sin\varphi\right)\right\}\delta w$$

$$-\iint zdsdz\left\{(X)\frac{\partial\delta w}{\partial x}+(Y)\frac{\partial\delta w}{\partial y}\right\}.$$

将板的体积元素表示为 $dfdz$，边缘表面元素表示为 $dsdz$，其中 ds 是中间表面的轮廓元素，并为了方便起见，用 w，x，y 替换 w_0，x_0，y_0。因此我们得到：

$$\delta P=\iint dfdz\left\{Z\delta w-z\left(X\frac{\partial\delta w}{\partial x}+Y\frac{\partial\delta w}{\partial y}\right)\right\}$$

$$+\iint dsdz\left\{(Z)\delta w-z\left((X)\frac{\partial\delta w}{\partial x}+(Y)\frac{\partial\delta w}{\partial y}\right)\right\}.$$

力 X，Y，Z 和压力 (X)，(Y)，(Z) 在这里可以视为与 w 无关，因为 w 是无穷小的。我们想要变换 δP 的表达式。它最初可以写成如下形式：

$$\delta P=\iiint dxdydz\left\{Z+z\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)\right\}\delta w$$

$$-\iiint zdxdydz\frac{\partial X\delta w}{\partial x}-\iiint zdxdydz\frac{\partial Y\delta w}{\partial y}$$

$$+\iint dsdz\left\{(Z)\delta w-z\left((X)\frac{\partial\delta w}{\partial x}+(Y)\frac{\partial\delta w}{\partial y}\right)\right\}.$$

这四个积分中的第二和第三个可以通过应用公式进行变换：

$$\begin{cases} \iint dxdy \frac{\partial F}{\partial x} = -\oint ds \cos\varphi F, \\ \iint dxdy \frac{\partial F}{\partial y} = -\oint ds \sin\varphi F. \end{cases}$$

这里 F 表示 x 和 y 的任意函数；双重积分是对有限表面进行的，单重积分是对同一表面的轮廓进行的；φ 表示从有限表面内部指向的法线与正 x-轴之间的角度，当它旋转到与正 x-轴平行的方向时（直到必须旋转使得在旋转 90° 后它占据正 y-轴的位置）。通过使用这些公式，对于 δP 得到以下值：

$$\delta P = \iiint dxdydz \left\{ Z + z\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\right\} \delta w$$

$$+ \iint dsdz \left\{ (Z) + z\left((X)\cos\varphi + (Y)\sin\varphi\right)\right\} \delta w$$

$$- \iint zdsdz \left\{ (X)\frac{\partial \delta w}{\partial x} + (Y)\frac{\partial \delta w}{\partial y}\right\}.$$

In the last three parts of δP, replace $\frac{\partial\delta w}{\partial x}$ and $\frac{\partial\delta w}{\partial y}$ with the differential quotients of δw along the inward normal of the contour, $\frac{\partial\delta w}{\partial N}$, and the differential quotients of δw along the arc of the contour, $\frac{\partial\delta w}{\partial s}$. Assume the arc s increases in such a way that the angle formed by the tangent in the direction of the increasing arc with the positive x-axis, and described in the manner defined for φ, until it is parallel to the tangent, is $\varphi-90^{\circ}$. The following equations then result:

$$\begin{cases}\dfrac{\partial\delta w}{\partial x}=\dfrac{\partial\delta w}{\partial N}\cos\varphi+\dfrac{\partial\delta w}{\partial s}\sin\varphi,\\ \dfrac{\partial\delta w}{\partial y}=\dfrac{\partial\delta w}{\partial N}\sin\varphi-\dfrac{\partial\delta w}{\partial s}\cos\varphi.\end{cases}$$

Thus, we have:

$$\begin{aligned}&\iint zdsdz\left\{(X)\frac{\partial\delta w}{\partial x}+(Y)\frac{\partial\delta w}{\partial y}\right\}\\ &=\iint zdsdz\left\{(X)\cos\varphi+(Y)\sin\varphi\right\}\frac{\partial\delta w}{\partial N}\\ &+\iint zdsdz\left\{(X)\sin\varphi-(Y)\cos\varphi\right\}\frac{\partial\delta w}{\partial s}.\end{aligned}$$

The second integral on the right side of this equation, we integrate partially with respect to s; the term emerging from the integral sign disappears because the integration refers to a closed curve; then substitute the left side of the equation into the expression for δP. This gives:

$$\begin{aligned}\delta P=&\iiint dxdydz\left\{Z+z\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)\right\}\delta w\\ &+\iint dsdz\left\{(Z)+z\frac{\partial\left((X)\sin\varphi-(Y)\cos\varphi\right)}{\partial s}+z\left(X\cos\varphi+Y\sin\varphi\right)\right\}\delta w\\ &-\iint dsdzz\left\{(X)\cos\varphi+(Y)\sin\varphi\right\}\frac{\partial\delta w}{\partial N}.\end{aligned}$$

Now form the second part of the right side of equation (9.). We consider a plane through a point of the middle surface, which has coordinates x, y, w, parallel to the z-axis and forming an angle θ with the xz-plane; ρ be the radius of curvature of the section of this plane.

在 δP 的最后三个部分中，将 $\frac{\partial \delta w}{\partial x}$ 和 $\frac{\partial \delta w}{\partial y}$ 替换为沿着轮廓的内法线方向的 δw 的微分商 $\frac{\partial \delta w}{\partial N}$，以及沿着轮廓弧长方向的 δw 的微分商 $\frac{\partial \delta w}{\partial s}$。假设弧长 s 按照这样的方式增加：即由沿着增加方向的切线与正 x-轴形成的角，并按照定义 φ 的方式旋转，直到它与切线平行时，该角为 $\varphi - 90^\circ$。则得到以下方程：

$$\begin{cases} \dfrac{\partial \delta w}{\partial x} = \dfrac{\partial \delta w}{\partial N} \cos \varphi + \dfrac{\partial \delta w}{\partial s} \sin \varphi, \\ \dfrac{\partial \delta w}{\partial y} = \dfrac{\partial \delta w}{\partial N} \sin \varphi - \dfrac{\partial \delta w}{\partial s} \cos \varphi. \end{cases}$$

因此，我们有：

$$\begin{aligned} &\iint z ds dz \left\{ (X) \frac{\partial \delta w}{\partial x} + (Y) \frac{\partial \delta w}{\partial y} \right\} \\ &= \iint z ds dz \left\{ (X) \cos \varphi + (Y) \sin \varphi \right\} \frac{\partial \delta w}{\partial N} \\ &+ \iint z ds dz \left\{ (X) \sin \varphi - (Y) \cos \varphi \right\} \frac{\partial \delta w}{\partial s}. \end{aligned}$$

此等式右边的第二个积分，我们对 s 进行部分积分；从积分符号中出现的项消失，因为积分指的是一个闭合曲线；然后将等式的左边代入 δP 的表达式。这给出：

$$\begin{aligned} \delta P = &\iiint dx dy dz \left\{ Z + z \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) \right\} \delta w \\ &+ \iint ds dz \left\{ (Z) + z \frac{\partial \left((X) \sin \varphi - (Y) \cos \varphi \right)}{\partial s} + z \left(X \cos \varphi + Y \sin \varphi \right) \right\} \delta w \\ &- \iint ds dz z \left\{ (X) \cos \varphi + (Y) \sin \varphi \right\} \frac{\partial \delta w}{\partial N}. \end{aligned}$$

现在形成方程 (9.) 右边的第二部分。我们考虑通过中间表面的一个点（其坐标为 x, y, w）的平面，该平面平行于 z-轴并与 xz-平面形成角度 θ；ρ 是该平面截面的曲率半径。

And the middle surface for the point (x, y, w), so it is

$$\frac{1}{q} = \frac{\partial^2 w}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 w}{\partial x \partial y}\cos\theta\sin\theta + \frac{\partial^2 w}{\partial y^2}\sin^2\theta.$$

The values of $\frac{1}{q_1}$ and $\frac{1}{q_2}$ are the maximum and minimum of $\frac{1}{q}$; they are therefore the roots of the equation

$$\left(\frac{\partial^2 w}{\partial x^2} - \lambda\right)\left(\frac{\partial^2 w}{\partial y^2} - \lambda\right) - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = 0.$$

From this follows:

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2},$$

$$\frac{1}{q_1^2} + \frac{1}{q_2^2} = \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2.$$

These values are to be substituted into equation (9.). Set

$$Q = \iint dxdy \left(\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right),$$

$$R = \iint dxdy \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2,$$

so equation (9.) becomes

$$\delta P - \tfrac{2}{3}\epsilon^2 K\left(\delta Q + \frac{\theta}{1+\theta}\delta R\right) = 0.$$

Now the formation of δQ and δR is necessary. It is found that

$$\delta Q = 2\iint dxdy \left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2} + 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \delta w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \delta w}{\partial y^2}\right\}.$$

However, it is:

$$\iint dxdy \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \delta w}{\partial x^2}$$

$$= \iint dxdy \frac{\partial}{\partial x}\frac{\partial^2 w}{\partial x^2}\frac{\partial \delta w}{\partial x} - \iint dxdy \frac{\partial}{\partial x}\frac{\partial^3 w}{\partial x^3}\delta w + \iint dxdy \frac{\partial^4 w}{\partial x^4}\delta w,$$

$$\iint dxdy \frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \delta w}{\partial x \partial y}$$

$$= \iint dxdy \frac{\partial}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\frac{\partial \delta w}{\partial y} - \iint dxdy \frac{\partial}{\partial y}\frac{\partial^2 w}{\partial x \partial y}\frac{\partial \delta w}{\partial x} + \iint dxdy \frac{\partial^4 w}{\partial x^2 \partial y^2}\delta w,$$

and also

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These four equations are to be added. On the left side of the resulting equation, one then obtains $\frac{1}{2}\delta Q$; on the right side of the same, one transforms those integrals which under the integral signs contain the element of the surface with a differential quotient taken with respect to x or y, using the formula (10.) in integrals that refer to the contour of the middle surface. In part of these integrals, the differential quotients $\frac{\partial \delta w}{\partial x}$ and $\frac{\partial \delta w}{\partial y}$ appear; these are expressed with the help of the equations (11.) through $\frac{\partial \delta w}{\partial N}$ and $\frac{\partial \delta w}{\partial s}$ and the terms containing $\frac{\partial \delta w}{\partial s}$ are integrated partially with respect to s. The terms that appear before the integral signs disappear because the integration refers to a closed curve, and it results that:

$$\delta Q = 2\iint dxdy \left\{ \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right\} \delta w$$

$$+2\oint ds \left\{ \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \cos\varphi + \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \sin\varphi \right.$$

$$\left. - \frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial x \partial y}(\cos^2\varphi - \sin^2\varphi) + \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right) \cos\varphi \sin\varphi \right) \right.$$

$$-2\oint ds \left\{ \frac{\partial^3 w}{\partial x^2}\cos^2\varphi + 2\frac{\partial^3 w}{\partial x \partial y}\cos\varphi\sin\varphi + \frac{\partial^3 w}{\partial y^2}\sin^2\varphi \right\} \frac{\partial \delta w}{\partial N} \Bigg\} .$$

Furthermore,

$$\delta R = 2\iint dxdy\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 \delta w}{\partial x^2}+\frac{\partial^2 \delta w}{\partial y^2}\right).$$

However, if F and G denote two arbitrary functions of x and y:

$$\iint dxdyF\left(\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}\right)$$

$$=\iint dxdyG\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right)+\oint ds\frac{\partial F}{\partial N}G-\oint dsF\frac{\partial G}{\partial N};$$

这四个方程需要相加。在得到的方程的左边，然后得到 $\frac{1}{2}\delta Q$；在同一个方程的右边，将那些包含对 x 或 y 的微分商的表面元素的积分转换为中间面的轮廓积分，使用公式 (10.)。在这些积分的一部分中，出现微分商 $\frac{\partial\delta w}{\partial x}$ 和 $\frac{\partial\delta w}{\partial y}$；这些通过方程 (11.) 用 $\frac{\partial\delta w}{\partial N}$ 和 $\frac{\partial\delta w}{\partial s}$ 表达，并且含有 $\frac{\partial\delta w}{\partial s}$ 的项部分地关于 s 积分。出现在积分符号前的项消失，因为积分指的是一个闭合曲线，结果是：

$$\delta Q = 2\iint dxdy\left\{\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right\}\delta w$$

$$+2\oint ds\left\{\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x\partial y^2}\right)\cos\varphi+\left(\frac{\partial^3 w}{\partial x^2\partial y}+\frac{\partial^3 w}{\partial y^3}\right)\sin\varphi\right.$$

$$-\frac{\partial}{\partial s}\left(\frac{\partial^2 w}{\partial x\partial y}(\cos^2\varphi-\sin^2\varphi)+\left(\frac{\partial^2 w}{\partial y^2}-\frac{\partial^2 w}{\partial x^2}\right)\cos\varphi\sin\varphi\right)$$

$$-2\oint ds\left\{\frac{\partial^3 w}{\partial x^2}\cos^2\varphi+2\frac{\partial^3 w}{\partial x\partial y}\cos\varphi\sin\varphi+\frac{\partial^3 w}{\partial y^2}\sin^2\varphi\right\}\frac{\partial\delta w}{\partial N}\Bigg\}.$$

此外，

$$\delta R = 2\iint dxdy\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 \delta w}{\partial x^2}+\frac{\partial^2 \delta w}{\partial y^2}\right).$$

然而，如果 F 和 G 表示 x 和 y 的任意函数：

$$\iint dxdyF\left(\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}\right)$$

$$=\iint dxdyG\left(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}\right)+\oint ds\frac{\partial F}{\partial N}G-\oint dsF\frac{\partial G}{\partial N};$$

One also sets
$F = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \quad G = \partial w$
and considers that
$\frac{\partial F}{\partial N} = \frac{\partial F}{\partial x} \cos\varphi + \frac{\partial F}{\partial y} \sin\varphi$
which follows from the equations (11.); thus, one obtains
$\delta R = 2 \iint dxdy \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) \delta w$
$+2 \oint ds \left(\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \cos\varphi + \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \sin\varphi \right) \delta w$
$-2 \oint ds \left(\frac{\partial^3 w}{\partial x^2} + \frac{\partial^3 w}{\partial y^2} \right) \frac{\partial \delta w}{\partial N}.$

Using the equations (12., 16., and 18.), one now forms the equation (14.); the left side of this can be represented as the sum of three integrals, of which the first is extended over the middle surface itself, the other two over the contour of the same, and of which the first two have under the integral signs the factor δw, the last the factor $\frac{\partial \delta w}{\partial N}$. According to the principles of the calculus of variations, the quantities with which δw and $\frac{\partial \delta w}{\partial N}$ must multiply must disappear. One thus obtains the partial differential equation
$\int_{-\epsilon}^{+\epsilon} Zdz + \frac{\partial}{\partial x} \int_{-\epsilon}^{+\epsilon} Xzdz + \frac{\partial}{\partial y} \int_{-\epsilon}^{+\epsilon} Yzdz$
$= \frac{4}{3} \epsilon^3 K \frac{1+2\theta}{1+\theta} \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right)$
and the two boundary conditions
$\int_{-\epsilon}^{+\epsilon} (Z) dz + \frac{\partial}{\partial s} \left(\sin\varphi \int_{-\epsilon}^{+\epsilon} (X) zdz - \cos\varphi \int_{-\epsilon}^{+\epsilon} (Y) zdz \right)$
$+ \cos\varphi \int_{-\epsilon}^{+\epsilon} Xzdz + \sin\varphi \int_{-\epsilon}^{+\epsilon} Yzdz$
$= \frac{4}{3} \epsilon^3 K \left\{ \frac{1+2\theta}{1+\theta} \left(\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \cos\varphi + \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \sin\varphi \right) \right.$
$\left. - \frac{\partial}{\partial s} \left(\frac{\partial^3 w}{\partial x \partial y} (\cos^2\varphi - \sin^2\varphi) + \left(\frac{\partial^3 w}{\partial y^2} - \frac{\partial^3 w}{\partial x^2} \right) \cos\varphi \sin\varphi \right) \right\},$

因此也设

$$F = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \quad G = \partial w$$

并注意到

$$\frac{\partial F}{\partial N} = \frac{\partial F}{\partial x}\cos\varphi + \frac{\partial F}{\partial y}\sin\varphi$$

这由方程 (11.) 得出；因此，得到

$$\delta R = 2\iint dxdy\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right)\delta w$$
$$+2\oint ds\left(\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\partial y^2}\right)\cos\varphi + \left(\frac{\partial^3 w}{\partial x^2\partial y} + \frac{\partial^3 w}{\partial y^3}\right)\sin\varphi\right)\delta w$$
$$-2\oint ds\left(\frac{\partial^3 w}{\partial x^2} + \frac{\partial^3 w}{\partial y^2}\right)\frac{\partial \delta w}{\partial N}.$$

利用方程 (12., 16., 和 18.)，现在形成方程 (14.)；其左侧可以表示为三个积分的和，其中第一个积分扩展到中间面本身，另外两个积分扩展到该面的轮廓，并且前两个积分符号下的因子是 δw，最后一个因子是 $\frac{\partial \delta w}{\partial N}$。根据变分法的原则，与 δw 和 $\frac{\partial \delta w}{\partial N}$ 相乘的量必须消失。因此得到偏微分方程

$$\int_{-\epsilon}^{+\epsilon} Zdz + \frac{\partial}{\partial x}\int_{-\epsilon}^{+\epsilon} Xzdz + \frac{\partial}{\partial y}\int_{-\epsilon}^{+\epsilon} Yzdz$$
$$= \frac{4}{3}\epsilon^3 K\frac{1+2\theta}{1+\theta}\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right)$$

以及两个边界条件

$$\int_{-\epsilon}^{+\epsilon}(Z)dz + \frac{\partial}{\partial s}\left(\sin\varphi\int_{-\epsilon}^{+\epsilon}(X)zdz - \cos\varphi\int_{-\epsilon}^{+\epsilon}(Y)zdz\right)$$
$$+\cos\varphi\int_{-\epsilon}^{+\epsilon} Xzdz + \sin\varphi\int_{-\epsilon}^{+\epsilon} Yzdz$$
$$= \frac{4}{3}\epsilon^3 K\left\{\frac{1+2\theta}{1+\theta}\left(\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x\partial y^2}\right)\cos\varphi + \left(\frac{\partial^3 w}{\partial x^2\partial y} + \frac{\partial^3 w}{\partial y^3}\right)\sin\varphi\right)\right.$$
$$\left.-\frac{\partial}{\partial s}\left(\frac{\partial^3 w}{\partial x\partial y}(\cos^2\varphi - \sin^2\varphi) + \left(\frac{\partial^3 w}{\partial y^2} - \frac{\partial^3 w}{\partial x^2}\right)\cos\varphi\sin\varphi\right)\right\},$$

Equation (21.)

$$\cos\varphi\int_{-\epsilon}^{+\epsilon}(X)zdz+\sin\varphi\int_{-\epsilon}^{+\epsilon}(Y)zdz$$
$$=\frac{4}{3}\epsilon^3K\left\{\frac{\theta}{1+\theta}\left(\frac{\partial^2w}{\partial x^2}+\frac{\partial^2w}{\partial y^2}\right)+\frac{\partial^2w}{\partial x^2}\cos^2\varphi+2\frac{\partial^2w}{\partial x\partial y}\cos\varphi\sin\varphi+\frac{\partial^2w}{\partial y^2}\sin^2\varphi\right\}.$$

Equation (19.) agrees with the partial differential equation derived by Poisson, except that Poisson set $\theta=\frac{1}{2}$, while here θ is left undetermined. The three boundary conditions of Poisson can be represented by equations (20., 21.) and the equation

$$\sin\varphi\int_{-\epsilon}^{+\epsilon}(X)zdz-\cos\varphi\int_{-\epsilon}^{+\epsilon}(Y)zdz$$
$$=-\frac{4}{3}\epsilon^3K\left(\frac{\partial^2w}{\partial x\partial y}(\cos^2\varphi-\sin^2\varphi)+\left(\frac{\partial^2w}{\partial y^2}-\frac{\partial^2w}{\partial x^2}\right)\cos\varphi\sin\varphi\right).$$

I will now show that w is determined up to an additive linear function of x and y, which remains arbitrary, by the equations (19., 20., 21.). From this it follows that the Poisson's equations are only satisfied in special cases where the given forces are such that equation (22.) is automatically fulfilled as soon as equations (19., 20., 21.) are satisfied.

Let w_1 and w_2 be two functions that, instead of w, satisfy the equations (19., 20., 21.); then $w_1-w_2=w$ satisfies the equations that arise from the above-mentioned ones when the left-hand sides are set to 0. I will prove that these equations are only satisfied by a linear function of x and y.

Consider the value of

$$\delta Q+\frac{\theta}{1+\theta}\delta R$$

formed once with the help of equations (15. and 17.), then with the help of equations (16. and 18.), and the two expressions obtained thereby are set equal to each other. In the identical equation thus obtained, let $\delta w=iw$, where under i an infinitely small constant is understood. If one then removes the factor $2i$ found therein, the equation becomes:

方程 (21.)
$\cos\varphi\int_{-\epsilon}^{+\epsilon}(X)zdz+\sin\varphi\int_{-\epsilon}^{+\epsilon}(Y)zdz$
$=\frac{4}{3}\epsilon^3K\left\{\frac{\theta}{1+\theta}\left(\frac{\partial^2w}{\partial x^2}+\frac{\partial^2w}{\partial y^2}\right)+\frac{\partial^2w}{\partial x^2}\cos^2\varphi+2\frac{\partial^2w}{\partial x\partial y}\cos\varphi\sin\varphi+\frac{\partial^2w}{\partial y^2}\sin^2\varphi\right\}.$

方程 (19.) 与 Poisson 导出的偏微分方程一致，除了 Poisson 设定 $\theta=\frac{1}{2}$，而这里 θ 保持未确定。Poisson 的三个边界条件可以通过方程 (20., 21.) 和方程
$\sin\varphi\int_{-\epsilon}^{+\epsilon}(X)zdz-\cos\varphi\int_{-\epsilon}^{+\epsilon}(Y)zdz$
$=-\frac{4}{3}\epsilon^3K\left(\frac{\partial^2w}{\partial x\partial y}(\cos^2\varphi-\sin^2\varphi)+\left(\frac{\partial^2w}{\partial y^2}-\frac{\partial^2w}{\partial x^2}\right)\cos\varphi\sin\varphi\right).$

我现在将证明 w 由方程 (19., 20., 21.) 确定到 x 和 y 的一个加性线性函数，该函数保持任意。由此得出，Poisson 的方程仅在给定力满足方程 (22.) 自动成立的情况下才成立。

设 w_1 和 w_2 是两个函数，它们代替 w 满足方程 (19., 20., 21.)；则 $w_1-w_2=w$ 满足从上述方程中得到的方程，当左端设为 0 时。我将证明这些方程仅由 x 和 y 的线性函数满足。

考虑值
$\delta Q+\frac{\theta}{1+\theta}\delta R$
一次用方程 (15. 和 17.) 帮助形成，然后用方程 (16. 和 18.) 帮助形成，并且由此得到的两个表达式相等。在由此得到的同一方程中，设 $\delta w=iw$，其中 i 被理解为无穷小的常数。如果去掉其中找到的因子 $2i$，方程变为：

$$\iint dxdy\left\{\left(\frac{\partial^3w}{\partial x^3}\right)^2+2\left(\frac{\partial^3w}{\partial x\partial y}\right)^2+\left(\frac{\partial^3w}{\partial y^3}\right)^2+\frac{\theta}{1+\theta}\left(\frac{\partial^3w}{\partial x^3}+\frac{\partial^3w}{\partial y^3}\right)^2\right\}$$

$$=\iint dxdy\frac{1+2\theta}{1+\theta}\left(\frac{\partial^4w}{\partial x^4}+2\frac{\partial^4w}{\partial x^2\partial y^2}+\frac{\partial^4w}{\partial y^4}\right)w$$

$$+\oint ds\left\{\frac{1+2\theta}{1+\theta}\left(\frac{\partial^3w}{\partial x^3}+\frac{\partial^3w}{\partial x\partial y^2}\right)\cos\varphi+\left(\frac{\partial^3w}{\partial x^2\partial y}+\frac{\partial^3w}{\partial y^3}\right)\sin\varphi\right\}w$$

$$-\oint ds\left\{\frac{\theta}{1+\theta}\left(\frac{\partial^3w}{\partial x\partial y}(\cos^2\varphi-\sin^2\varphi)+\left(\frac{\partial^3w}{\partial y^2}-\frac{\partial^3w}{\partial x^2}\right)\cos\varphi\sin\varphi\right)\right.$$

$$\left.-\frac{\partial}{\partial s}\left(\frac{\partial^3w}{\partial x\partial y}(\cos^2\varphi-\sin^2\varphi)+\left(\frac{\partial^3w}{\partial y^2}-\frac{\partial^3w}{\partial x^2}\right)\cos\varphi\sin\varphi\right)\right\}w$$

$$-\oint ds\left\{\frac{\theta}{1+\theta}\left(\frac{\partial^3w}{\partial x^2}+\frac{\partial^3w}{\partial y^2}\right)\cos^2\varphi+2\frac{\partial^3w}{\partial x\partial y}\cos\varphi\sin\varphi+\frac{\partial^3w}{\partial y^2}\sin^2\varphi\right\}\frac{\partial w}{\partial N}.$$

If w satisfies the equations into which equations (19., 20., 21.) pass when their left parts are set to 0, then the right side of this equation vanishes; hence the left side also vanishes. However, since θ is positive, it consists of a sum of purely positive quantities: all these quantities must therefore vanish for themselves, and thus for all points of the middle surface of the plate the equations

$$\frac{\partial^3 w}{\partial x^3} = 0, \quad \frac{\partial^3 w}{\partial x \partial y} = 0, \quad \frac{\partial^3 w}{\partial y^3} = 0$$

must be satisfied. These equations can only be satisfied by a linear function of x and y.

I will now apply the equations (19., 20., 21.) from the previous paragraph to derive the laws of vibrations of a free, circular disk. From these, we obtain for the vibrations of an arbitrarily shaped disk the partial differential equation:

$$(1.) \quad 0 = \varrho \frac{\partial^3 w}{\partial t^2} + \frac{2}{3} \frac{1+2\theta}{1+\theta} \epsilon^2 K \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right),$$

in which ϱ denotes the density of the disk, along with the boundary conditions:

$$(2.) \quad \begin{cases} 0 = \frac{1+2\theta}{1+\theta} \left(\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \cos\varphi + \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \sin\varphi \right) \\ \qquad - \frac{\partial}{\partial s} \left(\frac{\partial^2 w}{\partial x \partial y} (\cos^2\varphi - \sin^2\varphi) + \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right) \cos\varphi \sin\varphi \right), \\ 0 = \frac{\theta}{1+\theta} \left(\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial^2 w}{\partial x^2} \cos^2\varphi + 2 \frac{\partial^2 w}{\partial x \partial y} \cos\varphi \sin\varphi + \frac{\partial^2 w}{\partial y^2} \sin^2\varphi \right). \end{cases}$$

Man erhält diese Gleichungen, indem man in jenen $(X) = (Y) = (Z) = 0$,

$$\iint dxdy \left\{ \left(\frac{\partial^3 w}{\partial x^3} \right)^2 + 2 \left(\frac{\partial^3 w}{\partial x \partial y} \right)^2 + \left(\frac{\partial^3 w}{\partial y^3} \right)^2 + \frac{\theta}{1+\theta} \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial y^3} \right)^2 \right\}$$

$$= \iint dxdy \frac{1+2\theta}{1+\theta} \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) w$$

$$+ \oint ds \left\{ \frac{1+2\theta}{1+\theta} \left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \cos\varphi + \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \sin\varphi \right\} w$$

$$- \oint ds \left\{ \frac{\theta}{1+\theta} \left(\frac{\partial^3 w}{\partial x \partial y} (\cos^2\varphi - \sin^2\varphi) + \left(\frac{\partial^3 w}{\partial y^2} - \frac{\partial^3 w}{\partial x^2} \right) \cos\varphi \sin\varphi \right) \right.$$

$$\left. - \frac{\partial}{\partial s} \left(\frac{\partial^3 w}{\partial x \partial y} (\cos^2\varphi - \sin^2\varphi) + \left(\frac{\partial^3 w}{\partial y^2} - \frac{\partial^3 w}{\partial x^2} \right) \cos\varphi \sin\varphi \right) \right\} w$$

$$- \oint ds \left\{ \frac{\theta}{1+\theta} \left(\frac{\partial^3 w}{\partial x^2} + \frac{\partial^3 w}{\partial y^2} \right) \cos^2\varphi + 2 \frac{\partial^3 w}{\partial x \partial y} \cos\varphi \sin\varphi + \frac{\partial^3 w}{\partial y^2} \sin^2\varphi \right\} \frac{\partial w}{\partial N}.$$

如果 w 满足方程 (19., 20., 21.) 的左端设为 0 时的方程，则此方程的右端消失；因此左端也消失。然而，由于 θ 是正的，它由纯正量的和组成：所有这些量必须自行消失，并且因此对于板中间表面的所有点，方程

$$\frac{\partial^3 w}{\partial x^3} = 0, \quad \frac{\partial^3 w}{\partial x \partial y} = 0, \quad \frac{\partial^3 w}{\partial y^3} = 0$$

必须被满足。这些方程只能通过 x 和 y 的线性函数来满足。

§. 4.

我现在将应用前一段中的方程 (19., 20., 21.) 来推导自由圆形薄板振动的定律。从这些方程中，我们得到任意形状薄板振动的偏微分方程：

(1.) $0 = \varrho \frac{\partial^3 w}{\partial t^2} + \frac{2}{3} \frac{1+2\theta}{1+\theta} \epsilon^2 K \left(\frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right),$

其中 ϱ 表示薄板的密度，以及边界条件：

$$(2.) \quad \begin{cases} 0 = \frac{1+2\theta}{1+\theta} \left(\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \cos\varphi + \left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \sin\varphi \right) \\ \qquad - \frac{\partial}{\partial s} \left(\frac{\partial^2 w}{\partial x \partial y} (\cos^2\varphi - \sin^2\varphi) + \left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2} \right) \cos\varphi \sin\varphi \right), \\ 0 = \frac{\theta}{1+\theta} \left(\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial^2 w}{\partial x^2} \cos^2\varphi + 2 \frac{\partial^2 w}{\partial x \partial y} \cos\varphi \sin\varphi + \frac{\partial^2 w}{\partial y^2} \sin^2\varphi \right). \end{cases}$$

Man erhält diese Gleichungen, indem man in jenen $(X) = (Y) = (Z) = 0,$

设 $X=-\varphi\frac{\partial^2 u}{\partial t^2}$，$Y=-\varphi\frac{\partial^2 v}{\partial t^2}$，$Z=-\varphi\frac{\partial^2 w}{\partial t^2}$，并考虑到在（§ 2.）中所做的两个假设，即 $\frac{\partial^2 u}{\partial t^2}$ 和 $\frac{\partial^2 v}{\partial t^2}$ 不可能无限大到与 $\frac{\partial^2 w}{\partial t^2}$ 相比。除了条件（1. 和 2.），还需要补充一点，即当 $t=0$ 时，w 和 $\frac{\partial w}{\partial t}$ 应过渡为 x 和 y 的两个给定函数；这样 w 就完全确定了。

我们首先寻找微分方程（1）的一个特解，该解满足方程（2）。然后可以将其推广，使其也满足 $t=0$ 时的条件。

为了简化起见，设
$\frac{2}{3}\frac{1+2\theta}{1+\theta}\varepsilon^2\frac{K}{\rho}=a^2$
以及
$$(3.)\quad w=u\sin(4\lambda^2 at);$$
其中 u 是 x 和 y 的函数，λ 是一个常数，我们保留对它的选择权。通过（3.)，如果 u 满足以下方程，则方程（1.）将被满足：
$$(4.)\quad 16\lambda^4 u=\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}.$$

这可以通过以下两个方程替换：
$$(5.)\begin{cases}4\lambda^2 v=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\\4\lambda^2 u=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}.\end{cases}$$

若设
$$(6.)\quad u=S+D,\quad v=S-D,$$
则对于 S 和 D 可得以下微分方程：
$$4\lambda^2 S=\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2},$$
$$-4\lambda^2 D=\frac{\partial^2 D}{\partial x^2}+\frac{\partial^2 D}{\partial y^2}.$$

现在引入极坐标 r,ψ 替换直角坐标；则最后的方程变为：
$$4\lambda^2 S=\frac{\partial^2 S}{\partial r^2}+\frac{1}{r}\frac{\partial S}{\partial r}+\frac{1}{r^2}\frac{\partial^2 S}{\partial\psi^2},$$
$$-4\lambda^2 D=\frac{\partial^2 D}{\partial r^2}+\frac{1}{r}\frac{\partial D}{\partial r}+\frac{1}{r^2}\frac{\partial^2 D}{\partial\psi^2}.$$

Set $X = -\varphi \frac{\partial^2 u}{\partial t^2}$, $Y = -\varphi \frac{\partial^2 v}{\partial t^2}$, $Z = -\varphi \frac{\partial^2 w}{\partial t^2}$, and consider that according to the two assumptions made in (§ 2.), $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 v}{\partial t^2}$ cannot be infinitely large compared to $\frac{\partial^2 w}{\partial t^2}$. In addition to conditions (1. and 2.), it is necessary to add that for $t = 0$, w and $\frac{\partial w}{\partial t}$ should transition into two given functions of x and y; then w will be completely determined.

We first seek a particular solution of the differential equation (1) that satisfies equations (2). This can then be generalized so that it also meets the conditions valid for $t = 0$.

For brevity, set

$\frac{2}{3} \frac{1+2\theta}{1+\theta} \varepsilon^2 \frac{K}{\rho} = a^2$

and

$(3.) \quad w = u \sin(4\lambda^2 at);$

where u is a function of x and y, λ is a constant, over which we reserve the right to choose. Through (3.), equation (1.) will be satisfied if u fulfills the following equation:

$(4.) \quad 16\lambda^4 u = \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}.$

This can be replaced by the two equations

$(5.) \begin{cases} 4\lambda^2 v = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \\ 4\lambda^2 u = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}. \end{cases}$

If we set

$(6.) \quad u = S + D, \quad v = S - D,$

then for S and D the differential equations follow:

$4\lambda^2 S = \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2},$

$-4\lambda^2 D = \frac{\partial^2 D}{\partial x^2} + \frac{\partial^2 D}{\partial y^2}.$

Now introduce polar coordinates r, ψ in place of the rectangular coordinates; then the last equations become:

$4\lambda^2 S = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r} \frac{\partial S}{\partial r} + \frac{1}{r^2} \frac{\partial^2 S}{\partial \psi^2},$

$-4\lambda^2 D = \frac{\partial^2 D}{\partial r^2} + \frac{1}{r} \frac{\partial D}{\partial r} + \frac{1}{r^2} \frac{\partial^2 D}{\partial \psi^2}.$

72 4. Kirchhoff，关于弹性盘的平衡和运动。如果设

$$(7.)\begin{cases} S = A\cos n\psi \cdot X, \\ D = B\cos n\psi \cdot Y, \end{cases}$$

其中 A 和 B 是任意常数，n 是一个整数，而 X 和 Y 是 r 的两个函数，它们满足方程

$$(8.)\begin{cases} \frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} - \left(\frac{n^2}{r^2} + 4\lambda^2\right)X = 0, \\ \frac{d^2Y}{dr^2} + \frac{1}{r}\frac{dY}{dr} - \left(\frac{n^2}{r^2} - 4\lambda^2\right)Y = 0. \end{cases}$$

若引入 $x = \lambda r$，则这些方程变为

$$(9.)\begin{cases} \frac{d^2X}{dx^2} + \frac{1}{x}\frac{dX}{dx} - \left(\frac{n^2}{x^2} + 4\right)X = 0, \\ \frac{d^2Y}{dx^2} + \frac{1}{x}\frac{dY}{dx} - \left(\frac{n^2}{x^2} - 4\right)Y = 0. \end{cases}$$

其特解为：

$$(10.)\begin{cases} X^{(n)} = \frac{x^n}{1\cdot2\cdot3\cdots n}\left(1 + \frac{x^2}{1\cdot n+1} + \frac{x^4}{1\cdot2\cdot n+1\cdot n+2} + \frac{x^6}{1\cdot2\cdot3\cdot n+1\cdot n+2\cdot n+3} + \text{etc.}\right), \\ Y^{(n)} = \frac{x^n}{1\cdot2\cdot3\cdots n}\left(1 - \frac{x^2}{1\cdot n+1} + \frac{x^4}{1\cdot2\cdot n+1\cdot n+2} - \frac{x^6}{1\cdot2\cdot3\cdot n+1\cdot n+2\cdot n+3} + \text{etc.}\right), \end{cases}$$

以及其他特解：

$$(11.)\begin{cases} X^{(n)'} = X^{(n)}\int_{x_0}^{x}\frac{dx}{xX^{(n)}X^{(n)}}, \\ Y^{(n)'} = Y^{(n)}\int_{x_0}^{x}\frac{dx}{xY^{(n)}Y^{(n)}}; \end{cases}$$

其中 x_0 表示任意有限值。因此，方程（9.）的一般解为：

$$\begin{cases} X = \alpha X^{(n)} + \alpha' X^{(n)'}, \\ Y = \beta Y^{(n)} + \beta' Y^{(n)'}. \end{cases}$$

从方程（11.）可以看出，当 $x = 0$ 时，$X^{(n)'}$ 和 $Y^{(n)'}$ 变为无穷大；假设盘是完整的，不是环形的，则当 $r = 0$，即 $x = 0$ 时，u 和 v 以及 X 和 Y 必须保持有限，因此 α' 和 β' 必须消失。在不损害一般性的情况下，可以将常数 α 和 β 设为 1，因为我们在方程（7.）中已经引入了常数 A 和 B。

 This is satisfied if we set

$$(7.)\begin{cases} S = A\cos n\psi \cdot X, \\ D = B\cos n\psi \cdot Y, \end{cases}$$

where A and B are arbitrary constants, n is an integer, and X and Y are two functions of r that satisfy the equations

$$(8.)\begin{cases} \frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} - \left(\frac{n^2}{r^2} + 4\lambda^2\right) X = 0, \\ \frac{d^2Y}{dr^2} + \frac{1}{r}\frac{dY}{dr} - \left(\frac{n^2}{r^2} - 4\lambda^2\right) Y = 0. \end{cases}$$

If we introduce $x = \lambda r$, then these equations become

$$(9.)\begin{cases} \frac{d^2X}{dx^2} + \frac{1}{x}\frac{dX}{dx} - \left(\frac{n^2}{x^2} + 4\right) X = 0, \\ \frac{d^2Y}{dx^2} + \frac{1}{x}\frac{dY}{dx} - \left(\frac{n^2}{x^2} - 4\right) Y = 0. \end{cases}$$

The particular integrals are as follows:

$$(10.)\begin{cases} X^{(n)} = \frac{x^n}{1\cdot 2\cdot 3\cdots n}\left(1 + \frac{x^2}{1\cdot n+1} + \frac{x^4}{1\cdot 2\cdot n+1\cdot n+2} + \frac{x^6}{1\cdot 2\cdot 3\cdot n+1\cdot n+2\cdot n+3} + \text{etc.}\right), \\ Y^{(n)} = \frac{x^n}{1\cdot 2\cdot 3\cdots n}\left(1 - \frac{x^2}{1\cdot n+1} + \frac{x^4}{1\cdot 2\cdot n+1\cdot n+2} - \frac{x^6}{1\cdot 2\cdot 3\cdot n+1\cdot n+2\cdot n+3} + \text{etc.}\right), \end{cases}$$

and other particular integrals:

$$(11.)\begin{cases} X^{(n)'} = X^{(n)}\int_{x_0}^{x} \frac{dx}{xX^{(n)}X^{(n)}}, \\ Y^{(n)'} = Y^{(n)}\int_{x_0}^{x} \frac{dx}{xY^{(n)}Y^{(n)}}; \end{cases}$$

where x_0 denotes any finite quantity. The general integrals of equations (9.) are therefore

$$\begin{cases} X = \alpha X^{(n)} + \alpha' X^{(n)'}, \\ Y = \beta Y^{(n)} + \beta' Y^{(n)'}. \end{cases}$$

From equations (11.), it is evident that $X^{(n)'}$ and $Y^{(n)'}$ become infinite for $x = 0$; assuming the disk is complete and not ring-shaped, then for $r = 0$, i.e., $x = 0$, u and v, and thus also X and Y, must remain finite, and therefore α' and β' must vanish. The constants α and β can be set to 1 without loss of generality, since we have already introduced the constants A and B in equations (7.).

因此，我们用以下方程代替方程（7.）:

$$(12.)\begin{cases} S = A\cos n\psi \cdot X^{(n)}, \\ D = B\cos n\psi \cdot Y^{(n)}, \end{cases}$$

其中 $X^{(n)}$ 和 $Y^{(n)}$ 是由方程（10.）确定的函数。我注意到，$Y^{(n)}$ 是贝塞尔引入记号 I_{2x}^{n} 的函数。

现在我们将寻找常数 A、B 和 λ 的值，使得方程（2.）得到满足。在这些方程中，弧长 s 在与方程（11.）（§ 3.）所指方向相同的方向上被视为增加。从那里给出的定义可以看出，如果让 ψ 在该方向上增加，并选择 s 的起点，使得

$s = l\psi,$

其中 l 表示盘的半径，

$\varphi = \psi + 180^\circ.$

利用这一点，通过引入极坐标 r 和 ψ 而不是直角坐标，方程（2.）将具有以下形式:

$$(13.)\begin{cases} \frac{1+2\theta}{1+\theta}\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \psi^2}\right) + \frac{1}{r^2}\frac{\partial}{\partial \psi}\left(\frac{\partial^2 w}{\partial r \partial \psi} - \frac{1}{r}\frac{\partial w}{\partial \psi}\right) = 0, \\ \frac{\theta}{1+\theta}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \psi^2}\right) + \frac{\partial^2 w}{\partial r^2} = 0. \end{cases}$$

这些方程必须在 $r = l$ 处对所有 ψ 的值和所有 t 的值都成立。根据（3.），当用 u 替换 w 时，也必须存在从（13.）得出的方程。考虑到

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \psi^2} = 4\lambda^2 v,$

这是方程（5.）的第一部分所说的:

$4\lambda^2\frac{1+2\theta}{1+\theta}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^3 u}{\partial r \partial \psi^2} - \frac{1}{r^3}\frac{\partial^2 u}{\partial \psi^2} = 0;$

$4\lambda^2\frac{\theta}{1+\theta}v + \frac{\partial^2 u}{\partial r^2} = 0.$

将 u 和 v 的值代入此处，这些值由（6. 和 12.）得出，用 $\frac{x}{\lambda}$ 替换 r，并表示 u 的二阶导数。

Therefore, we write instead of equations (7.):

$$(12.)\begin{cases} S = A\cos n\psi \cdot X^{(n)}, \\ D = B\cos n\psi \cdot Y^{(n)}, \end{cases}$$

where $X^{(n)}$ and $Y^{(n)}$ are the functions determined by equations (10.). I note that $Y^{(n)}$ is the function for which Bessel introduced the notation I_{2x}^{n}.

We will now seek to determine the constants A, B, and λ such that equations (2.) are satisfied. In these equations, the arc length s is considered increasing in the direction specified in equations (11.) (§ 3.). From the definition given there, it follows that if ψ increases in that direction and the starting point of s is chosen so that

$s = l\psi,$

where l denotes the radius of the disk,

$\varphi = \psi + 180^\circ.$

Using this, the equations (2.) take the following form when polar coordinates r and ψ are introduced instead of rectangular coordinates:

$$(13.)\begin{cases} \frac{1+2\theta}{1+\theta}\frac{\partial}{\partial r}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \psi^2}\right)+\frac{1}{r^2}\frac{\partial}{\partial \psi}\left(\frac{\partial^2 w}{\partial r\partial \psi}-\frac{1}{r}\frac{\partial w}{\partial \psi}\right)=0, \\ \frac{\theta}{1+\theta}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \psi^2}\right)+\frac{\partial^2 w}{\partial r^2}=0. \end{cases}$$

These equations must be satisfied for $r=l$, for all values of ψ, and for all values of t. According to (3.), the equations derived from (13.) when u is written instead of w must also hold. These equations give, considering that

$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \psi^2}=4\lambda^2 v,$

which is what the first equation of (5.) states:

$4\lambda^2\frac{1+2\theta}{1+\theta}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^3 u}{\partial r\partial \psi^2}-\frac{1}{r^3}\frac{\partial^2 u}{\partial \psi^2}=0;$

$4\lambda^2\frac{\theta}{1+\theta}v+\frac{\partial^2 u}{\partial r^2}=0.$

Substituting here the values of u and v obtained from (6. and 12.), replacing r with $\frac{x}{\lambda}$, and expressing the second derivatives of u.

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通过函数 $X^{(n)}$ 和 $Y^{(n)}$ 及其一阶导数，并最终使 $\frac{1+2\theta}{1+\theta}=\gamma$，得到以下结果：

$$\begin{cases} 0=A\left\{n^2X^{(n)}-x(n^2-4\gamma x^2)\frac{dX^{(n)}}{dx}\right\}+B\left\{n^2Y^{(n)}-x(n^2+4\gamma x)\frac{dY^{(n)}}{dx}\right\}, \\ 0=A\left\{(n^2+4\gamma x^2)X^{(n)}-x\frac{dX^{(n)}}{dx}\right\}+B\left\{(n^2-4\gamma x^2)Y^{(n)}-x\frac{dY^{(n)}}{dx}\right\}. \end{cases}$$

这些方程需要在 $r=l$ 即 $x=\lambda l$ 处满足。为此，它们的行列式必须为零，因为 A 和 B 不应等于零。因此对于 $x=\lambda l$ 必须有：

$$\begin{aligned} 0= & 8\gamma n^2x^2X^{(n)}Y^{(n)}-8\gamma n^2x^3\left(X^{(n)}\frac{dY^{(n)}}{dx}+Y^{(n)}\frac{dX^{(n)}}{dx}\right) \\ & -(n^2(n^2-1)x+16\gamma^2x^5)\left(X^{(n)}\frac{dY^{(n)}}{dx}-Y^{(n)}\frac{dX^{(n)}}{dx}\right) \\ & +8\gamma x^4\frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx}. \end{aligned}$$

从这个方程确定 λ 并从两个方程（14）中的一个确定 A 和 B 的比例，就满足条件（2）。我们将方程（15）的一个根记为 $\lambda_{n\mu}$，并定义：

$$U_{n\mu} = X^{(n)}\left\{(n^2-4\gamma x^2)Y^{(n)} - x\frac{dY^{(n)}}{dx}\right\}\Big|_{x=\lambda_{n\mu}l} - Y^{(n)}\left\{(n^2+4\gamma x^2)X^{(n)} - x\frac{dX^{(n)}}{dx}\right\}\Big|_{x=\lambda_{n\mu}l},$$

则通过 $w = C_{n\mu}\sin(4\lambda_{n\mu}^2 at)\cos n\psi U_{n\mu}$ 或者

$$w = \left\{\cos(4\lambda_{n\mu}^2 at)(A_{n\mu}\cos n\psi + B_{n\mu}\sin n\psi) + \sin(4\lambda_{n\mu}^2 at)(C_{n\mu}\cos n\psi + D_{n\mu}\sin n\psi)\right\}U_{n\mu}$$

满足方程（1）和（2），其中 $A_{n\mu}, B_{n\mu}, C_{n\mu}, D_{n\mu}$ 是任意常数。

现在我们更详细地研究方程（15），首先将等式的右边展开成正幂级数......

Using the functions $X^{(n)}$ and $Y^{(n)}$ and their first derivatives, and finally making $\frac{1+2\theta}{1+\theta} = \gamma$, we obtain:

$$\begin{cases} 0 = A\left\{n^2X^{(n)} - x(n^2-4\gamma x^2)\frac{dX^{(n)}}{dx}\right\} + B\left\{n^2Y^{(n)} - x(n^2+4\gamma x)\frac{dY^{(n)}}{dx}\right\}, \\ 0 = A\left\{(n^2+4\gamma x^2)X^{(n)} - x\frac{dX^{(n)}}{dx}\right\} + B\left\{(n^2-4\gamma x^2)Y^{(n)} - x\frac{dY^{(n)}}{dx}\right\}. \end{cases}$$

These equations must be satisfied for $r = l$, i.e., for $x = \lambda l$. For this, their determinant must vanish since A and B should not be zero. Therefore, for $x = \lambda l$:

$$\begin{aligned} 0 =& 8\gamma n^2 x^2 X^{(n)}Y^{(n)} - 8\gamma n^2 x^3\left(X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx}\right) \\ &- (n^2(n^2-1)x + 16\gamma^2 x^5)\left(X^{(n)}\frac{dY^{(n)}}{dx} - Y^{(n)}\frac{dX^{(n)}}{dx}\right) \\ &+ 8\gamma x^4\frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx}. \end{aligned}$$

Determine λ from this equation and the ratio of A and B from one of the two equations (14), so that the conditions (2) are satisfied. We denote a root of equation (15) by $\lambda_{n\mu}$ and define:

$$U_{n\mu} = X^{(n)} \left\{ (n^2 - 4\gamma x^2) Y^{(n)} - x \frac{dY^{(n)}}{dx} \right\} \Big|_{x=\lambda_{n\mu} l} - Y^{(n)} \left\{ (n^2 + 4\gamma x^2) X^{(n)} - x \frac{dX^{(n)}}{dx} \right\} \Big|_{x=\lambda_{n\mu} l},$$

then the equations (1) and (2) are satisfied by $w = C_{n\mu} \sin(4\lambda_{n\mu}^2 at) \cos n\psi U_{n\mu}$ or also by

$$w = \left\{ \cos(4\lambda_{n\mu}^2 at)(A_{n\mu} \cos n\psi + B_{n\mu} \sin n\psi) + \sin(4\lambda_{n\mu}^2 at)(C_{n\mu} \cos n\psi + D_{n\mu} \sin n\psi) \right\} U_{n\mu},$$

where $A_{n\mu}, B_{n\mu}, C_{n\mu}, D_{n\mu}$ are arbitrary constants.

Now we want to examine equation (15) more closely and first develop the right-hand side into a series of positive powers...

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随着 x 的增加，我们必须首先形成 $X^{(n)}Y^{(n)}$ 的乘积。然而，在此过程中可以直接避免无限级数（10.）的直接相乘。实际上，通过使用微分方程（9.），其中 $X^{(n)}$ 和 $Y^{(n)}$ 满足这些方程，可以找到一个四阶线性微分方程来表示 $X^{(n)}Y^{(n)}$，并由此确定该乘积，同时考虑到它显然应该具有的形式。设
$H = X^{(n)}Y^{(n)}$
并对该方程进行反复微分以得到 H 关于 x 的前四个导数的值。然后，利用（9.）中的函数及其一阶导数来表示 $X^{(n)}$ 和 $Y^{(n)}$ 的二阶及更高阶导数。这样，我们得到五个方程，它们分别表示 H 及其四个一阶导数作为四个量的线性齐次函数：
$X^{(n)}Y^{(n)}, \quad X^{(n)}\frac{dY^{(n)}}{dx}, \quad Y^{(n)}\frac{dX^{(n)}}{dx}, \quad \frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx}.$

将这五个方程中的第一个与（1.）相乘，其余的依次与不确定系数 A_1, A_2, A_3, A_4 相乘，然后将所有方程相加，并确定这些系数，使得在总和中这四个量的因子消失；于是得到恒等式：
(18.) $\quad 0 = H + A_1 \frac{dH}{dx} + A_2 \frac{d^2H}{dx^2} + A_3 \frac{d^3H}{dx^3} + A_4 \frac{d^4H}{dx^4}.$

如果我们将表示 $X^{(n)}$ 和 $Y^{(n)}$ 的二阶及更高阶导数通过这些量本身及其一阶导数表达的方程写成如下形式：
$\frac{d^2X^{(n)}}{dx^2} = a_2 X^{(n)} + a_2' \frac{dX^{(n)}}{dx}, \quad \frac{d^2Y^{(n)}}{dx^2} = b_2 Y^{(n)} + b_2' \frac{dY^{(n)}}{dx},$
$\frac{d^3X^{(n)}}{dx^3} = a_3 X^{(n)} + a_3' \frac{dX^{(n)}}{dx}, \quad \frac{d^3Y^{(n)}}{dx^3} = b_3 Y^{(n)} + b_3' \frac{dY^{(n)}}{dx},$
$\frac{d^4X^{(n)}}{dx^4} = a_4 X^{(n)} + a_4' \frac{dX^{(n)}}{dx}, \quad \frac{d^4Y^{(n)}}{dx^4} = b_4 Y^{(n)} + b_4' \frac{dY^{(n)}}{dx},$

则对于系数 A 可以得到以下方程：
$1 + (a_2 + b_2)A_2 + (a_3 + b_3)A_3 + (a_4 + b_4 + 6a_2b_2)A_4 = 0,$
$A_1 + a_2' A_2 + (a_3' + 3b_2)A_3 + (a_4' + 4b_3 + 6a_2'b_2)A_4 = 0,$
$A_1 + b_2' A_2 + (b_3' + 3a_2)A_3 + (b_4' + 4a_3 + 6a_2b_2')A_4 = 0,$
$2A_2 + 3(a_2' + b_2')A_3 + (4a_3' + 4b_3' + 6a_2'b_2')A_4 = 0.$

将 a 和 b 的值代入这些方程。

As x progresses, we must first form the product $X^{(n)}Y^{(n)}$. However, in this process, the direct multiplication of the infinite series (10.) can be avoided. Indeed, by using the differential equations (9.), which $X^{(n)}$ and $Y^{(n)}$ satisfy, one can find a fourth-order linear differential equation for $X^{(n)}Y^{(n)}$ and determine this product while considering the form it apparently should have. Set

$H = X^{(n)}Y^{(n)}$

and form the values of the first four derivatives of H with respect to x through repeated differentiation of this equation. Then express the second and higher derivatives of $X^{(n)}$ and $Y^{(n)}$ using these functions themselves and their first derivatives with the help of (9.). This results in five equations that give H and its four first derivatives as linear homogeneous functions of the four quantities:

$X^{(n)}Y^{(n)}, \quad X^{(n)}\frac{dY^{(n)}}{dx}, \quad Y^{(n)}\frac{dX^{(n)}}{dx}, \quad \frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx}.$

Multiply the first of these five equations by (1.), and the others successively by the undetermined coefficients A_1, A_2, A_3, A_4, add all of them, and determine these coefficients so that the factors of those four quantities disappear in the sum; then the following identity is obtained:

$$(18.) \quad 0 = H + A_1\frac{dH}{dx} + A_2\frac{d^2H}{dx^2} + A_3\frac{d^3H}{dx^3} + A_4\frac{d^4H}{dx^4}.$$

If we write the equations through which the second and higher derivatives of $X^{(n)}$ and $Y^{(n)}$ are expressed by these quantities themselves and their first derivatives as follows:

$$\frac{d^2X^{(n)}}{dx^2} = a_2X^{(n)} + a_2'\frac{dX^{(n)}}{dx}, \quad \frac{d^2Y^{(n)}}{dx^2} = b_2Y^{(n)} + b_2'\frac{dY^{(n)}}{dx},$$
$$\frac{d^3X^{(n)}}{dx^3} = a_3X^{(n)} + a_3'\frac{dX^{(n)}}{dx}, \quad \frac{d^3Y^{(n)}}{dx^3} = b_3Y^{(n)} + b_3'\frac{dY^{(n)}}{dx},$$
$$\frac{d^4X^{(n)}}{dx^4} = a_4X^{(n)} + a_4'\frac{dX^{(n)}}{dx}, \quad \frac{d^4Y^{(n)}}{dx^4} = b_4Y^{(n)} + b_4'\frac{dY^{(n)}}{dx},$$

then the following equations result for the coefficients A:

$$1 + (a_2 + b_2)A_2 + (a_3 + b_3)A_3 + (a_4 + b_4 + 6a_2b_2)A_4 = 0,$$
$$A_1 + a_2'A_2 + (a_3' + 3b_2)A_3 + (a_4' + 4b_3 + 6a_2'b_2)A_4 = 0,$$
$$A_1 + b_2'A_2 + (b_3' + 3a_2)A_3 + (b_4' + 4a_3 + 6a_2b_2')A_4 = 0,$$
$$2A_2 + 3(a_2' + b_2')A_3 + (4a_3' + 4b_3' + 6a_2'b_2')A_4 = 0.$$

Substitute the values of a and b into these equations.

从（9.）得出的结果中解出这些方程，并将 A_1、A_2、A_3、A_4 的值代入方程（18.），则该方程变为：
$$0 = 64H + \frac{4n^2-4}{x^3}\frac{dH}{dx} - \frac{4n^2-4}{x^2}\frac{d^2H}{dx^2} + \frac{4}{x}\frac{d^3H}{dx^3} + \frac{d^4H}{dx^4}.$$

从（10.）可以得出：
$$(19.)\quad H = X^{(n)}Y^{(n)} = \frac{x^{2n}}{(1\cdot 2\cdot 3\cdots n)^2}\left(1 + B_1x^4 + B_2x^8 + B_3x^{12} + B_4x^{16} + \ldots\right)$$
必须如此；将这个级数代入上面找到的微分方程中，得到
$$B_k = -\frac{B_{k-1}}{k\cdot n+k\cdot n+2k-1\cdot n+2k},$$
因此
$$(20.)\quad B_k = \frac{(-1)^k}{1\cdot 2\cdots k\cdot n+1\cdot n+2\cdots n+k\cdot n+1\cdot n+2\cdots n+2k}.$$

为了形成方程（15.），还需要展开以下表达式：
$$X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx},\quad X^{(n)}\frac{dY^{(n)}}{dx} - Y^{(n)}\frac{dX^{(n)}}{dx},\quad \frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx}.$$

为此，我们使用以下与方程（9.）等价的方程：
$$\frac{dH}{dx} = X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx},$$
$$\frac{d^2H}{dx^2} = \frac{2n^2}{x^2}X^{(n)}Y^{(n)} - \frac{1}{x}\left(X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx}\right) + 2\frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx},$$
$$\frac{d^3H}{dx^3} = -\frac{6n^2}{x^3}X^{(n)}Y^{(n)} + \frac{4n^2+2}{x^2}\left(X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx}\right) + 8\left(X^{(n)}\frac{dY^{(n)}}{dx} - Y^{(n)}\frac{dX^{(n)}}{dx}\right) - \frac{6}{x}\frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx}.$$

通过解这些方程，我们得到：
$$X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx} = \frac{dH}{dx},$$
$$X^{(n)}\frac{dY^{(n)}}{dx} - Y^{(n)}\frac{dX^{(n)}}{dx} = \frac{d^3H}{dx^3} + \frac{3}{x}\frac{d^2H}{dx^2} - \frac{4n^2-4}{x^2}\frac{dH}{dx},$$
$$\frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx} = \frac{d^2H}{dx^2} + \frac{1}{x}\frac{dH}{dx} - \frac{2n^2}{x^2}H.$$

考虑方程（19. 和 20.），这三个量依次为：

From the results obtained from (9.), solve these equations and substitute the values of A_1, A_2, A_3, A_4 into equation (18.), then this equation becomes:
$0 = 64H + \frac{4n^2-4}{x^3}\frac{dH}{dx} - \frac{4n^2-4}{x^2}\frac{d^2H}{dx^2} + \frac{4}{x}\frac{d^3H}{dx^3} + \frac{d^4H}{dx^4}.$

From (10.), it follows that:
$(19.) \quad H = X^{(n)}Y^{(n)} = \frac{x^{2n}}{(1\cdot 2\cdot 3\cdots n)^2}\left(1 + B_1x^4 + B_2x^8 + B_3x^{12} + B_4x^{16} + \dots\right)$
must be so; substituting this series into the above differential equation, we find
$B_k = -\frac{B_{k-1}}{k\cdot n+k\cdot n+2k-1\cdot n+2k},$
thus
$(20.) \quad B_k = \frac{(-1)^k}{1\cdot 2\cdots k\cdot n+1\cdot n+2\cdots n+k\cdot n+1\cdot n+2\cdots n+2k}.$

To form equation (15.), we further need to expand the following expressions:
$X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx}, \quad X^{(n)}\frac{dY^{(n)}}{dx} - Y^{(n)}\frac{dX^{(n)}}{dx}, \quad \frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx}.$

For this purpose, we use the following equations, which are identical in consideration of equations (9.):
$\frac{dH}{dx} = X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx},$
$\frac{d^2H}{dx^2} = \frac{2n^2}{x^2}X^{(n)}Y^{(n)} - \frac{1}{x}\left(X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx}\right) + 2\frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx},$
$\frac{d^3H}{dx^3} = -\frac{6n^2}{x^3}X^{(n)}Y^{(n)} + \frac{4n^2+2}{x^2}\left(X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx}\right) + 8\left(X^{(n)}\frac{dY^{(n)}}{dx} - Y^{(n)}\frac{dX^{(n)}}{dx}\right) - \frac{6}{x}\frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx}.$

By solving these equations, we obtain:
$X^{(n)}\frac{dY^{(n)}}{dx} + Y^{(n)}\frac{dX^{(n)}}{dx} = \frac{dH}{dx},$
$X^{(n)}\frac{dY^{(n)}}{dx} - Y^{(n)}\frac{dX^{(n)}}{dx} = \frac{d^3H}{dx^3} + \frac{3}{x}\frac{d^2H}{dx^2} - \frac{4n^2-4}{x^2}\frac{dH}{dx},$
$\frac{dX^{(n)}}{dx}\frac{dY^{(n)}}{dx} = \frac{d^2H}{dx^2} + \frac{1}{x}\frac{dH}{dx} - \frac{2n^2}{x^2}H.$

Considering equations (19. and 20.), these three quantities are successively:

$$\frac{2x^{2n-1}}{(1\cdot 2\cdots n)^2}\left(n+\sum_{k=1}^{\infty}(n+2k)B_k x^{4k}\right),$$

$$\frac{4x^{2n-3}}{(1\cdot 2\cdots n)^2}\sum_{k=1}^{\infty}k(n+k)(n+2k)B_k x^{4k},$$

$$\frac{x^{2n-2}}{(1\cdot 2\cdots n)^2}\left(n^2+\sum_{k=1}^{\infty}(2(n+2k)^2-n^2)B_k x^{4k}\right).$$

接下来，如果省略因子 x^{2n+2}，则方程（15.）变为：
(21.)　$0=(4\gamma-1)n^2(n-1)+\sum_{k=1}^{\infty}(-1)^k\frac{E_k}{M_k}x^{4k}$,
其中
$E_k=-n^2(n^2-1)+4\gamma(n+2k)(n+2k+1)(n(n-1)-2k+4\gamma k(n+k))$,
$M_k=1\cdot 2\cdots k\cdot n+1\cdot n+2\cdots n+k\cdot n+1\cdot n+2\cdots n+2k+1.$

可以看出，方程（21.）的根通过 l 除以得到的值是 $\lambda_{n\mu}$ 可以设置的值，这些值只包含 x 的幂次，其指数是 4 的倍数。因此，如果 $l\lambda$ 是其中一个根，则 $-l\lambda$、$l\lambda\sqrt{-1}$ 和 $-l\lambda\sqrt{-1}$ 也是它的根。现在我将证明方程（21.）的所有根的四次幂都是实数且为正；由此可以得出，在每组四个根中，如给定的一个根所示，必须存在一个实数正根。如前所述，θ 是一个正值；因此 γ，它被设为 $\frac{1+2\theta}{1+\theta}$，大于 1。由此得出 E_k 总是正的，因此方程（21.）右侧的项具有交替的符号。因此，不可能使该方程的任何根的四次幂为负。要证明它们也不能是虚数，可以通过以下间接方式来说明。

假设 λ 和 λ' 是方程（21.）的两个根。对于第一个根，由（16.）确定的函数 $U_{n\mu}=U$，对于第二个根为 U'：则可以证明
(22.)　$(\lambda^4-\lambda'^4)\int_0^l UU'rdr=0.$

我们将其视为已证明，并将证明 λ'' 不能是虚数。设
$\lambda=p+q\sqrt{-1}$,
$U=P+Q\sqrt{-1}.$

$$\frac{2x^{2n-1}}{(1\cdot 2\cdots n)^2}\left(n+\sum_{k=1}^{\infty}(n+2k)B_k x^{4k}\right),$$

$$\frac{4x^{2n-3}}{(1\cdot 2\cdots n)^2}\sum_{k=1}^{\infty}k(n+k)(n+2k)B_k x^{4k},$$

$$\frac{x^{2n-2}}{(1\cdot 2\cdots n)^2}\left(n^2+\sum_{k=1}^{\infty}(2(n+2k)^2-n^2)B_k x^{4k}\right).$$

Next, if the factor x^{2n+2} is omitted, equation (15.) becomes:
$(21.)\quad 0=(4\gamma-1)n^2(n-1)+\sum_{k=1}^{\infty}(-1)^k\frac{E_k}{M_k}x^{4k}$,
where
$E_k=-n^2(n^2-1)+4\gamma(n+2k)(n+2k+1)(n(n-1)-2k+4\gamma k(n+k))$,
$M_k=1\cdot 2\cdots k\cdot n+1\cdot n+2\cdots n+k\cdot n+1\cdot n+2\cdots n+2k+1.$

It can be seen that the roots of equation (21.), when divided by l, are the values that can be set for $\lambda_{n\mu}$, containing only powers of x whose exponents are multiples of 4. It follows that if $l\lambda$ is one of its roots, then $-l\lambda$, $l\lambda\sqrt{-1}$, and $-l\lambda\sqrt{-1}$ are also its roots. I will now show that the fourth powers of all roots of equation (21.) are real and positive; from this it will follow that in every group of four roots, as given, there must be a real positive root. As previously noted, θ is a positive quantity; thus γ, which was set to $\frac{1+2\theta}{1+\theta}$, is greater than 1. From this it follows that E_k is always positive, so the terms on the right side of equation (21.) have alternating signs. Therefore, it is impossible for the fourth power of any root of this equation to be negative. To show that they cannot also be imaginary, the following indirect method can be used.

Let λ and λ' be two roots of equation (21.). For the first root, the function $U_{n\mu}=U$ determined by (16.), and for the second root, U': then it can be shown that
$(22.)\quad (\lambda^4-\lambda'^4)\int_0^l UU'rdr=0.$

We assume this has been proven and will show that λ'' cannot be imaginary. Let
$\lambda=p+q\sqrt{-1}$,
$U=P+Q\sqrt{-1}.$

方程（21）的另一个根必须是 $(p-q\sqrt{-1})l$。我们设 $\lambda'=p-q\sqrt{-1}$，则有 $U'=P-Q\sqrt{-1}$。方程（22）变为：

$$pq(p^2-q^2)\int_0^l (P^2+Q^2)rdr=0.$$

这里出现的积分不能为零，因为它是一系列正数之和，因此必须有：

$$pq(p^2-q^2)=0,$$

这是 λ^4 为实数的条件。

现在我们要证明方程（22）的正确性。由此可知，在（17）中给出的 w 的表达式满足方程（1），可以得出，如果设：

$$\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}-\frac{n^2}{r^2}U=4\lambda^2V,$$

同时也有：

$$\begin{cases}\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}-\frac{n^2}{r^2}U=4\lambda^2V,\\ \frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}-\frac{n^2}{r^2}V=4\lambda^2U.\end{cases}$$

由此可知，同样的 w 表达式也满足方程（2）或方程（13），即当 $r=l$ 时，

$$\begin{cases}\frac{1+2\theta}{1+\theta}\frac{d}{dr}\left(\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}-\frac{n^2}{r^2}U\right)-\frac{n^2}{r^2}\left(\frac{dU}{dr}-\frac{1}{r}U\right)=0,\\ \frac{\theta}{1+\theta}\left(\frac{d^2U}{dr^2}+\frac{1}{r}\frac{dU}{dr}-\frac{n^2}{r^2}U\right)+\frac{d^2U}{dr^2}=0.\end{cases}$$

方程（23）可以写成：

$$4\lambda^2V=r^{n-1}\frac{d}{dr}\frac{1}{r^{2n-1}}\frac{d}{dr}r^nU,$$
$$4\lambda^2U=r^{n-1}\frac{d}{dr}\frac{1}{r^{2n-1}}\frac{d}{dr}r^nV.$$

通过将第一个方程中的 V 值代入第二个方程，得到：

$$16\lambda^4U=r^{n-1}\frac{d}{dr}\frac{1}{r^{2n-1}}\frac{d}{dr}r^{2n-1}\frac{d}{dr}\frac{1}{r^{2n-1}}\frac{d}{dr}r^nU.$$

用同样的方法，可以得到一个方程，其中用 U' 和 λ' 替换 U 和 λ。然后将 V 的值代入……

Another root of equation (21) must then be $(p - q\sqrt{-1})l$. We set $\lambda' = p - q\sqrt{-1}$, then $U' = P - Q\sqrt{-1}$. Equation (22) becomes:

$$pq(p^2 - q^2)\int_0^l (P^2 + Q^2) r dr = 0.$$

The integral here cannot vanish because it is a sum of positive quantities; therefore, it must be:

$$pq(p^2 - q^2) = 0,$$

and this is the condition for λ^4 to be real.

Now we want to prove the correctness of equation (22). From this, it follows that the expression given in (17) for w satisfies equation (1), which means if we set:

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \frac{n^2}{r^2}U = 4\lambda^2 V,$$

then simultaneously:

$$\begin{cases} \frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \frac{n^2}{r^2}U = 4\lambda^2 V, \\ \frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{n^2}{r^2}V = 4\lambda^2 U. \end{cases}$$

From this, it follows that the same expression for w satisfies equations (2) or equations (13), which are the same, so for $r = l$,

$$\begin{cases} \frac{1+2\theta}{1+\theta}\frac{d}{dr}\left(\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \frac{n^2}{r^2}U\right) - \frac{n^2}{r^2}\left(\frac{dU}{dr} - \frac{1}{r}U\right) = 0, \\ \frac{\theta}{1+\theta}\left(\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \frac{n^2}{r^2}U\right) + \frac{d^2U}{dr^2} = 0. \end{cases}$$

Equations (23) can be written as:

$$4\lambda^2 V = r^{n-1}\frac{d}{dr}\frac{1}{r^{2n-1}}\frac{d}{dr}r^n U,$$
$$4\lambda^2 U = r^{n-1}\frac{d}{dr}\frac{1}{r^{2n-1}}\frac{d}{dr}r^n V.$$

By substituting the value of V from the first of these two equations into the second, we obtain:

$$16\lambda^4 U = r^{n-1}\frac{d}{dr}\frac{1}{r^{2n-1}}\frac{d}{dr}r^{2n-1}\frac{d}{dr}\frac{1}{r^{2n-1}}\frac{d}{dr}r^n U.$$

In the same way, an equation can be obtained by replacing U and λ with U' and λ'. Then substitute the value of V from...

将 $16\lambda^4 U$ 从（25.）代入积分

$16\lambda^4 \int UU' r dr$

并进行四次部分积分，得到积分

$\int rdrU \cdot r^{n-1} \frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^{2n-1} \frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U',$

这等于

$= 16\lambda'^4 \int UU' r dr.$

通过上述方法，我们得到方程

$(26.) \quad (16\lambda^4 - \lambda'^4) \int UU' r dr = \frac{1}{r^{n-1}} U' \left(\frac{d}{dr} r^{2n-1} \frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U \right)$
$- \left(\frac{d}{dr} r^n U' \right) \left(\frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U \right)$
$+ \left(\frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U' \right) \left(\frac{d}{dr} r^n U \right)$
$- \left(\frac{d}{dr} r^{2n-1} \frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U' \right) \frac{1}{r^{n-1}} U.$

现在可以证明，该方程右侧的表达式在 $r = 0$ 和 $r = l$ 时都消失。当 $r = 0$ 时，它消失是显而易见的，因为 U 和 U' 的形式为

$cr^n + c_1 r^{n+2} + c_2 r^{n+4} + \cdots$

为了证明它在 $r = l$ 时也消失，我们应用方程（24.）。这些方程可以写成：

$\frac{d^2U}{dr^2} = \frac{\theta}{1+2\theta} \left(\frac{n^2}{r^2} U - \frac{1}{r} \frac{dU}{dr} \right),$
$\frac{d^3U}{dr^3} = -\frac{3n^2}{r^3} U + \frac{n^2+(n^2+1)(1+3\theta)}{(1+2\theta)r^2} \frac{dU}{dr}.$

借助这些方程以及当用 U' 替换 U 时产生的方程，我们可以用 U 和 U' 及其一阶导数表示 U 和 U' 的二阶和三阶导数；当我们执行（26.）中给出的微分时，会发现（26.）右侧的项相互抵消。因此，方程（22.）是正确的。

到目前为止，还没有考虑我们的偏微分方程在 $t = 0$ 时应满足的条件。我们现在将尝试推广已找到的解（17.），使其也能满足这些条件。从表达式开始，

Substituting $16\lambda^4 U$ from (25.) into the integral

$16\lambda^4 \int UU' r dr$

and integrating by parts four times, we obtain the integral

$\int rdrU \cdot r^{n-1} \frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^{2n-1} \frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U',$

which is equal to

$= 16\lambda'^4 \int UU' r dr.$

Following this method, we get the equation

$(26.) \quad (16\lambda^4 - \lambda'^4) \int UU' r dr = \frac{1}{r^{n-1}} U' \left(\frac{d}{dr} r^{2n-1} \frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U \right)$
$- \left(\frac{d}{dr} r^n U' \right) \left(\frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U \right)$
$+ \left(\frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U' \right) \left(\frac{d}{dr} r^n U \right)$
$- \left(\frac{d}{dr} r^{2n-1} \frac{d}{dr} \frac{1}{r^{2n-1}} \frac{d}{dr} r^n U' \right) \frac{1}{r^{n-1}} U.$

To prove that it also vanishes for $r = l$, we apply equations (24.). These equations can be written as:
$\frac{d^2U}{dr^2} = \frac{\theta}{1+2\theta}\left(\frac{n^2}{r^2}U - \frac{1}{r}\frac{dU}{dr}\right)$,
$\frac{d^3U}{dr^3} = -\frac{3n^2}{r^3}U + \frac{n^2+(n^2+1)(1+3\theta)}{(1+2\theta)r^2}\frac{dU}{dr}$.

Using these equations and those derived from them when U' replaces U, we can express the second and third derivatives of U and U' in terms of U and U' and their first derivatives; when performing the differentiations given in (26.), we find that the terms on the right side of (26.) cancel each other out. Therefore, equation (22.) is correct.

So far, no consideration has been given to the conditions that our partial differential equation should satisfy at $t = 0$. We will now seek to generalize the found solution (17.) so that it also satisfies these conditions. Starting from the expression,

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在（17.）中设 w 的值，可以对 μ 和 n 进行求和，并将这个双重求和设为 w。在第一个求和中，我们可以只考虑实数正根 $\lambda_{n\mu}$；通过考虑负数和虚数根 $\lambda_{n\mu}$，我们不会增加 w 表达式的普遍性；因为：如果 $\lambda = -\lambda'$，则有
$U_{n\mu} = U'$
如果 $\lambda = \lambda'\sqrt{-1}$，则有
$U = (-1)^{n+1}U'$.

现在我们将 $l\lambda_{n_0}, l\lambda_{n_1}, l\lambda_{n_2}, \ldots, l\lambda_{n_\mu}, \ldots$ 理解为方程（21.）的正实根，按大小顺序排列，使得 $l\lambda_{n_0}$ 是最小的，并设
$w = \sum_{n=0}^{\infty}\sum_{\mu=0}^{\infty}\{\cos(4\lambda_{n\mu}^2 at)(A_{n\mu}\cos n\psi + B_{n\mu}\sin n\psi)$
$+ \sin(4\lambda_{n\mu}^2 at)(C_{n\mu}\cos n\psi + D_{n\mu}\sin n\psi)\} U_{n\mu}$.

常数 A, B, C, D 必须这样确定，使得当 $t = 0$ 时，
$w = F(r, \psi)$,
$\frac{\partial w}{\partial t} = \Phi(r, \psi)$.

其中 F 和 Φ 是给定的 r 和 ψ 的函数。从第一个条件可以得到 A, B 的值，从第二个条件可以得到 C, D 的值；而且这些值的求法与前者完全相似；因此，只需展示如何找到前者即可。第一个条件要求
$F(r, \psi) = \sum_{n=0}^{\infty}\sum_{\mu=0}^{\infty}(A_{n\mu}\cos n\psi + B_{n\mu}\sin n\psi)U_{n\mu}$.

我们假设 $F(r, \psi)$ 可以展开为 ψ 的倍数的余弦和正弦项，即
$F(r, \psi) = F_0(r) + F_1(r)\cos\psi + F_2(r)\cos 2\psi + \cdots$
$+ F_1'(r)\sin\psi + F_2'(r)\sin 2\psi + \cdots$.

然后将这个展开式代入 $F(r, \psi)$。由此可以看出，

In (17.), where w is set, we can take the sum with respect to μ and with respect to n, and set this double sum equal to w. In the first of the two summations, we can restrict ourselves to considering the real positive values $\lambda_{n\mu}$; by considering the negative and imaginary values $\lambda_{n\mu}$, we do not gain any generality in the expression for w; because: if $\lambda = -\lambda'$, then
$U_{n\mu} = U'$,
and if $\lambda = \lambda'\sqrt{-1}$, then
$U = (-1)^{n+1} U'$.

Now let $l\lambda_{n_0}, l\lambda_{n_1}, l\lambda_{n_2}, \ldots, l\lambda_{n_\mu}, \ldots$ represent the positive real roots of equation (21.), arranged in order of their magnitude so that $l\lambda_{n_0}$ is the smallest, and let
$w = \sum_{n=0}^{\infty} \sum_{\mu=0}^{\infty} \left\{ \cos(4\lambda_{n\mu}^2 at)(A_{n\mu} \cos n\psi + B_{n\mu} \sin n\psi) \right.$
$\left. + \sin(4\lambda_{n\mu}^2 at)(C_{n\mu} \cos n\psi + D_{n\mu} \sin n\psi) \right\} U_{n\mu}.$

The constants A, B, C, D must now be determined such that for $t = 0$,
$w = F(r, \psi)$,
$\frac{\partial w}{\partial t} = \Phi(r, \psi)$,

where F and Φ are given functions of r and ψ. From the first condition, the values of A, B will be obtained, and from the second condition, the values of C, D; and these can be found in a similar manner as the former; therefore, it is sufficient to show how the former can be found. The first condition requires that
$F(r, \psi) = \sum_{n=0}^{\infty} \sum_{\mu=0}^{\infty} (A_{n\mu} \cos n\psi + B_{n\mu} \sin n\psi) U_{n\mu}.$

We assume that $F(r, \psi)$ can be expanded into cosine and sine terms of multiples of ψ, i.e.,
$F(r, \psi) = F_0(r) + F_1(r) \cos \psi + F_2(r) \cos 2\psi + \cdots$
$+ F_1'(r) \sin \psi + F_2'(r) \sin 2\psi + \cdots$.

Then substitute this expansion into $F(r, \psi)$. It then becomes clear that

$$(27.)\quad \begin{cases} F_n(r) = \sum_0^\infty A_{n\mu} U_{n\mu}, \\ F'_n(r) = \sum_0^\infty B_{n\mu} U_{n\mu} \end{cases}$$

函数 $F_n(r)$ 和 $F'_n(r)$ 可以视为已知；因此，确定大小 A 和 B 的问题可以归结为将给定的 r 函数展开为 $U_{n_0}, U_{n_1}, U_{n_2}, \ldots$ 的函数。假设这种展开是可能的，我们可以利用（22.）方程中表达的定理找到这些系数。该方程表明，如果 μ 和 μ' 是两个不同的数，则有：
$\int_0^l U_{n\mu} U_{n\mu'} r dr = 0$
因此，从方程（27.）得出：
$A_{n\mu} \int_0^l U_{n\mu} U_{n\mu} r dr = \int_0^l F_n(r) U_{n\mu} r dr,$
$B_{n\mu} \int_0^l U_{n\mu} U_{n\mu} r dr = \int_0^l F'_n(r) U_{n\mu} r dr.$

§. 5.

为了将理论与经验进行比较，研究盘子产生纯音的情况非常重要。不同音调的振动频率以及每种音调中存在的节点线，在这里作为主要的比较点。我们现在就来研究这种情况。在这种情况下，w 应通过以下表达式表示：
$(28.)\quad w = \left\{ \cos(4\lambda_{n\mu}^2 at)(A \cos n\psi + B \sin n\psi) \right.$
$\left. + \sin(4\lambda_{n\mu}^2 at)(C \cos n\psi + D \sin n\psi) \right\} U_{n\mu}.$

音调由 $\lambda_{n\mu}$ 确定，其振动频率，即单位时间内完成的振动次数为：
$\frac{4\lambda_{n\mu}^2 a}{\pi}.$

节点线是那些对于所有 t 值，$w = 0$ 的线；因此，它们包含满足方程
$(29.)\quad U_{n\mu} = 0$
的点。

$$(27.)\quad \begin{cases} F_n(r) = \sum_0^\infty A_{n\mu} U_{n\mu}, \\ F'_n(r) = \sum_0^\infty B_{n\mu} U_{n\mu} \end{cases}$$

The functions $F_n(r)$ and $F'_n(r)$ can be considered as given; thus, determining the values of A and B reduces to the task of expanding a given function of r into the functions $U_{n_0}, U_{n_1}, U_{n_2}, \ldots$. Assuming that this expansion is possible, we can find these coefficients using the theorem expressed by equation (22.). This equation shows that if μ and μ' are two different numbers, then:
$\int_0^l U_{n\mu} U_{n\mu'} r dr = 0$
Therefore, from equations (27.), it follows that:
$A_{n\mu} \int_0^l U_{n\mu} U_{n\mu} r dr = \int_0^l F_n(r) U_{n\mu} r dr,$
$B_{n\mu} \int_0^l U_{n\mu} U_{n\mu} r dr = \int_0^l F'_n(r) U_{n\mu} r dr.$

§. 5.

For comparing theory with experience, it is important to investigate the case where the vibrations of the disk produce a pure tone. The relationships of the vibration frequencies of the different tones that a disk can produce, and the nodal lines present in each individual case, serve here as the main points of comparison. We now want to deal with this case. In the same way, w must be represented by the following expression:

$$(28.)\quad w=\left\{\cos(4\lambda_{n\mu}^2 at)(A\cos n\psi+B\sin n\psi)\right.$$
$$\left.+\sin(4\lambda_{n\mu}^2 at)(C\cos n\psi+D\sin n\psi)\right\}U_{n\mu}.$$

The tone is determined by $\lambda_{n\mu}$ in such a way that its frequency, i.e., the number of vibrations completed in the unit of time, is:

$$\frac{4\lambda_{n\mu}^2 a}{\pi}.$$

The nodal lines are those lines for which, for all values of t, $w=0$; therefore, they contain the points for which either the equation

$$(29.)\quad U_{n\mu}=0$$

holds.

满足条件，或者两个方程

$$(30.)\quad \begin{cases} A\cos n\psi+B\sin n\psi=0, \\ C\cos n\psi+D\sin n\psi=0 \end{cases}$$

成立。方程（29.）提供了一些 r 的值作为根。它有多少个实数根小于 l，就有多少个与盘子边缘同心的圆出现在声图中；这些圆的数量和大小仅取决于音调，而与系数 A,B,C,D 的值无关。如果 $A:B\neq C:D$，则方程（30.）对任何点都不成立；在这种情况下，这些圆是唯一的节点线。如果存在这种比例关系，则方程（30.）给出 n 个 ψ 的值，其中每两个相邻的值相差 $\frac{\pi}{n}$；因此，除了这些圆之外，还有 n 条直径作为节点线，将盘子的边缘等分。

这些理论的一般结果在本质上与实验一致。实验表明，节点线由与盘子边缘同心的圆和将这些圆等分的直径组成，如果我们忽略某些变形，这些变形似乎主要是由于盘子不能像理论假设的那样完全自由振动造成的。实验还表明，在某些音调下，有时会出现直径作为节点线，有时又会缺失。当它们缺失时，撒在盘子上的沙子虽然也按直径排列，但在盘子运动过程中并不固定，而是振荡。如果要解释这一有趣的现象，我们必须追踪一粒沙子的运动，它从盘子的一个位置被抛到另一个位置，而盘子本身则按照方程（28.）所表示的运动进行。不过，我在这里不进一步讨论这一点。

Chladni 通过实验发现，具有相同数量直径的音调（即对应相同的 n 值的音调），其振动频率除了一种情况外，

It must be satisfied, or the two equations

$$(30.)\quad \begin{cases} A\cos n\psi + B\sin n\psi = 0, \\ C\cos n\psi + D\sin n\psi = 0 \end{cases}$$

hold. Equation (29.) provides certain values of r as roots. The number of real roots it has that are smaller than l is the same as the number of concentric circles with the periphery of the disk that appear in the sound figure; the number and size of these circles depend solely on the tone and are independent of the values of the coefficients A, B, C, D. Equations (30.) will not be satisfied for any point if $A : B \neq C : D$; in this case, these circles are the only nodal lines. If this proportion exists, then equations (30.) give n values of ψ, each two consecutive ones differing by $\frac{\pi}{n}$; thus, in addition to these circles, there are also n diameters as nodal lines which divide the periphery of the disk into equal parts.

These general results of the theory are essentially in agreement with experience. The experiment shows that the nodal lines consist of circles that are concentric with the periphery of the disk and diameters that divide these circles into equal parts, if we ignore certain distortions that these lines suffer and which, as it seems to me, mainly have their basis in the fact that the disk is not completely free as the theory assumes. The experiment also shows that at a certain tone, sometimes diameters occur as nodal lines, and sometimes they are missing. When they are missing, the sand scattered on the disk is arranged along the diameters; however, these do not remain fixed during the movement of the disk but oscillate. If one wanted to explain this interesting phenomenon, one would have to follow the movement of a grain of sand which, being thrown from one place on the disk to another, is hurled from one place to another while the disk itself performs the motion expressed by equation (28.). However, I will not go into this consideration here.

Chladni found through experiments that the frequencies of tones which have the same number of diameters in their sound figures (i.e., the tones corresponding to the same value of n), except for one case,

对于最深的情况，它与连续偶数或奇数的平方的关系取决于直径的数量是偶数还是奇数。我现在将证明理论也提供了相同的规律。这可以从方程（21.）的变形中得出。

对于由方程（10.）中的第二个方程确定的函数 $Y^{(0)}$，Poisson 开发了以下半收敛级数：

$$Y^{(0)} = \frac{1}{\sqrt{2\pi}}\frac{4}{\sqrt{x}}\left\{(\cos 2x + \sin 2x)\left(1 - \frac{(1.3)^2}{1.2}\frac{1}{(16x)^2} + \frac{(1.3.5.7)^2}{1.2.3.4}\frac{1}{(16x)^4} - \cdots\right)\right\}$$
$$+(\sin 2x - \cos 2x)\left(\frac{1^2}{4}\frac{1}{16x} - \frac{(1.3.5)^2}{1.2.3}\frac{1}{(16x)^3} + \frac{(1.3.5.7.9)^2}{1.2.3.4.5}\frac{1}{(16x)^5} - \cdots\right).$$

以类似的方式，可以找到

$$X^{(0)} = \frac{1}{2\sqrt{\pi}}\frac{e^{2x}}{\sqrt{x}}\left\{1 + \frac{1^2}{1}\frac{1}{16x} + \frac{(1.3)^2}{1.2}\frac{1}{(16x)^2} + \frac{(1.3.5)^2}{1.2.3}\frac{1}{(16x)^3} + \cdots\right\}.$$

从方程（10.）可以得到：

$$Y^{(n+1)} = -\frac{1}{2}x^n\frac{d}{dx}\left(\frac{Y^{(n)}}{x^n}\right),$$
$$X^{(n+1)} = \frac{1}{2}x^n\frac{d}{dx}\left(\frac{X^{(n)}}{x^n}\right).$$

如果在这里设 $n = 0$ 并用上面给出的级数代替 $Y^{(0)}, X^{(0)}$，则可以得到类似的级数 $Y^{(1)}, X^{(1)}$；从这些级数中又可以找到类似的级数 $Y^{(2)}, X^{(2)}$ 等等。结果如下：

$$Y^{(n)} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{x}}\left\{(\cos(2x-\tfrac{1}{2}n\pi)+\sin(2x-\tfrac{1}{2}n\pi))\left(1-\frac{(1-4n^2)(9-4n^2)}{1.2}\frac{1}{(16x)^2}+\frac{(1-4n^2)(9-4n^2)(25-4n^2)(49-4n^2)}{1.2.3.4}\frac{1}{(16x)^4}+\cdots\right.\right.$$
$$\left.\left.+(\sin(2x-\tfrac{1}{2}n\pi)-\cos(2x-\tfrac{1}{2}n\pi))\left(\frac{(1-4n^2)}{4}\frac{1}{16x}-\frac{(1-4n^2)(9-4n^2)(25-4n^2)}{1.2.3}\frac{1}{(16x)^3}+\cdots\right)\right\},$$
$$X^{(n)} = \frac{1}{2\sqrt{\pi}}\frac{e^{2x}}{\sqrt{x}}\left\{1+\frac{(1-4n^2)}{1}\frac{1}{16x}+\frac{(1-4n^2)(9-4n^2)}{1.2}\frac{1}{(16x)^2}+\frac{(1-4n^2)(9-4n^2)(25-4n^2)}{1.2.3}\frac{1}{(16x)^3}+\cdots\right\}.$$

这两个级数中的第一个在稍微不同的形式下已经由雅各比教授在舒马赫天文通讯第28卷第94页给出。如果将这些 $Y^{(n)}$ 和 $X^{(n)}$ 的值代入方程（15.），该方程与方程（21.）相同，则可以……

For the deepest case, it behaves similarly to the squares of consecutive even or odd numbers, depending on whether the number of diameters is even or odd. I will now show that the theory provides the same law. This can be seen from a transformation of equation (21.).

For the function $Y^{(0)}$, which is determined by the second of equations (10.), Poisson developed the following semi-convergent series:

$$Y^{(0)} = \frac{1}{\sqrt{2\pi}}\frac{4}{\sqrt{x}}\left\{(\cos 2x+\sin 2x)\left(1-\frac{(1.3)^2}{1.2}\frac{1}{(16x)^2}+\frac{(1.3.5.7)^2}{1.2.3.4}\frac{1}{(16x)^4}-\cdots\right)\right\}$$
$$+(\sin 2x-\cos 2x)\left(\frac{1^2}{4}\frac{1}{16x}-\frac{(1.3.5)^2}{1.2.3}\frac{1}{(16x)^3}+\frac{(1.3.5.7.9)^2}{1.2.3.4.5}\frac{1}{(16x)^5}-\cdots\right).$$

In a similar way, one finds

$$X^{(0)} = \frac{1}{2\sqrt{\pi}}\frac{e^{2x}}{\sqrt{x}}\left\{1+\frac{1^2}{1}\frac{1}{16x}+\frac{(1.3)^2}{1.2}\frac{1}{(16x)^2}+\frac{(1.3.5)^2}{1.2.3}\frac{1}{(16x)^3}+\cdots\right\}.$$

From equations (10.) it follows that:

$$Y^{(n+1)} = -\frac{1}{2}x^n\frac{d}{dx}\left(\frac{Y^{(n)}}{x^n}\right),$$
$$X^{(n+1)} = \frac{1}{2}x^n\frac{d}{dx}\left(\frac{X^{(n)}}{x^n}\right).$$

If we set $n = 0$ here and substitute the above series for $Y^{(0)}, X^{(0)}$, we obtain similar series for $Y^{(1)}, X^{(1)}$; from these, we again find similar series for $Y^{(2)}, X^{(2)}$, etc. The result is:

$$Y^{(n)} = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{x}}\left\{(\cos(2x-\tfrac{1}{2}n\pi)+\sin(2x-\tfrac{1}{2}n\pi))\left(1-\frac{(1-4n^2)(9-4n^2)}{1.2}\frac{1}{(16x)^2}+\frac{(1-4n^2)(9-4n^2)(25-4n^2)(49-4n^2)}{1.2.3.4}\frac{1}{(16x)^4}+\cdots\right.\right.$$
$$\left.\left.+(\sin(2x-\tfrac{1}{2}n\pi)-\cos(2x-\tfrac{1}{2}n\pi))\left(\frac{(1-4n^2)}{4}\frac{1}{16x}-\frac{(1-4n^2)(9-4n^2)(25-4n^2)}{1.2.3}\frac{1}{(16x)^3}+\cdots\right)\right\},$$
$$X^{(n)} = \frac{1}{2\sqrt{\pi}}\frac{e^{2x}}{\sqrt{x}}\left\{1+\frac{(1-4n^2)}{1}\frac{1}{16x}+\frac{(1-4n^2)(9-4n^2)}{1.2}\frac{1}{(16x)^2}+\frac{(1-4n^2)(9-4n^2)(25-4n^2)}{1.2.3}\frac{1}{(16x)^3}+\cdots\right\}.$$

The first of these two series has already been given in a slightly different form by Professor Jacobi in Schumacher's Astronomical Notices, Vol. 28, p. 94. If these values of $Y^{(n)}$ and $X^{(n)}$ are substituted into equation (15.), which is identical with equation (21.), then...

从结果方程 $\log(2x-\frac{1}{2}n\pi)$ 可以表示为两个级数的商，这两个级数按 $\frac{1}{16x}$ 的幂次递进。由此得到：

$$(31.)\quad \tan(2x-\tfrac{1}{2}n\pi)=\frac{\frac{\mathfrak{B}}{16x}+\frac{\mathfrak{C}}{(16x)^2}-\frac{\mathfrak{D}}{(16x)^3}+\cdots}{\mathfrak{A}+\frac{\mathfrak{B}}{16x}+\frac{\mathfrak{D}}{(16x)^3}+\cdots},$$

其中
$\mathfrak{A}=\gamma,$
$\mathfrak{B}=\gamma(1-4n^2)-8,$
$\mathfrak{C}=\gamma(1-4n^2)(9-4n^2)+48(1+4n^2),$
$\mathfrak{D}=-\gamma\frac{1}{3}((1-4n^2)(9-4n^2)(13-4n^2))+8(9+136n^2+80n^4).$

如果 x 很大，可以将方程 (31.) 右边设为 0；通过这种方式，可以得到该方程根的近似值，即表达式 $\frac{1}{4}\pi(n+2h)$ 所取的值，当 h 设为整数时。由此得出上面提到的、由 Chladni 发现的规律：音调的振动数与方程 (31.) 根的平方成正比。此外，该方程还表明，随着音调越高，振动数与连续偶数或奇数平方的比例关系越接近；它提供了一种方便的方法来确定属于 n 的所有音调（除了最低音），并以较高的精度进行确定。关于近似值中的 h 数值，数值计算显示它等于 μ；因此对于较大的 $\mu l_{n\mu}$ 值，其近似于 $\frac{1}{4}\pi(n+2\mu)$：这一结果与 Chladni 的观察完全一致。

为了确定一个 n 值的最低音调，必须计算方程 (21.) 的最小根。该方程可以通过除以 x^4 转换为以下形式：
$0=1-\frac{x^4}{A_1}+\frac{x^8}{A_2}-\frac{x^{12}}{A_3}+\text{etc.},$
其中 $n=0$ 和 $n=1$。计算给出 $\log A_1, \log A_2, \ldots$ 的值如下：

From the resulting equation $\log(2x-\frac{1}{2}n\pi)$, it can be expressed as the quotient of two series that progress according to the powers of $\frac{1}{16x}$. This yields:

$$(31.)\quad \tan(2x-\tfrac{1}{2}n\pi)=\frac{\frac{\mathfrak{B}}{16x}+\frac{\mathfrak{C}}{(16x)^2}-\frac{\mathfrak{D}}{(16x)^3}+\cdots}{\mathfrak{A}+\frac{\mathfrak{B}}{16x}+\frac{\mathfrak{D}}{(16x)^3}+\cdots},$$

where
$\mathfrak{A}=\gamma,$
$\mathfrak{B}=\gamma(1-4n^2)-8,$
$\mathfrak{C}=\gamma(1-4n^2)(9-4n^2)+48(1+4n^2),$
$\mathfrak{D}=-\gamma\frac{1}{3}((1-4n^2)(9-4n^2)(13-4n^2))+8(9+136n^2+80n^4).$

If x is large, the right side of equation (31.) can be set to 0; this way, one obtains approximate values for the roots of the same equation, which are the values taken by the expression $\frac{1}{4}\pi(n+2h)$ when h is set to whole numbers. From this, the above-mentioned law discovered by Chladni follows: that the vibration numbers of the tones are proportional to the squares of the roots of equation (31.). Furthermore, this equation shows that the proportionality of the vibration numbers with the squares of consecutive even or odd numbers becomes closer as the tones become higher; and it provides a means to conveniently determine all tones belonging to a value of n, except the lowest, with greater accuracy. Regarding the number h in the approximation value of l_n, numerical calculation shows that it equals μ; thus for large values of $\mu l_{n\mu}$, it is approximately $\frac{1}{4}\pi(n+2\mu)$: a result that is in complete agreement with Chladni's observations.

To determine the lowest tones for a value of n, one must calculate the smallest roots of equation (21.). This equation can be brought into the form:

$$0=1-\frac{x^4}{A_1}+\frac{x^8}{A_2}-\frac{x^{12}}{A_3}+\text{etc.},$$

by dividing it by x^4 for $n=0$ and $n=1$. The calculation gives the following values of $\log A_1, \log A_2, \ldots$:

对于 $\theta=\frac{1}{2}$，即 $\gamma=\frac{4}{3}$（根据 Poisson）:

n	0	1	2	3
$\log A_1$	0.664 2079	1.348 0266	0.265 0703	0.973 3073
$\log A_2$	2.535 1132	3.588 0591	2.061 5798	3.105 2465
$\log A_3$	5.097 3650	6.406 5347	4.588 9514	5.868 2055
$\log A_4$	8.149 924	9.655 978	7.620 8431	9.083 199
$\log A_5$	11.583 4	13.249 6	11.040 582	12.653 148
$\log A_6$	15.33	17.130	14.776 04	16.516 13
$\log A_7$	21.3	18.778 0	20.629 0	-
$\log A_8$	-	23.01	24.96	-
$\log A_9$	-	-	29.48	-

对于 $\theta = 1$，即 $\gamma = \frac{3}{2}$（根据 Wertheim）：

n	0	1	2	3
$\log A_1$	0.681 2413	1.352 1826	0.224 2682	0.939 3022
$\log A_2$	2.556 3026	3.594 0911	2.018 9332	3.066 4444
$\log A_3$	5.120 4304	6.413 6351	4.546 2236	5.827 9058
$\log A_4$	8.174 058	9.663 768	7.578 3037	9.042 324
$\log A_5$	11.608 3	13.257 8	10.998 263	12.612 037
$\log A_6$	15.35	17.138	14.733 92	16.474 93
$\log A_7$	21.3	18.736	20.587 7	-
$\log A_8$	-	22.97	24.92	-
$\log A_9$	-	-	29.44	-

由此得出以下 $\log(\lambda_{n\mu} l)^4$ 的值：

对于 $\theta = \frac{1}{2}$：

μ	$n = 0$	$n = 1$	$n = 2$	$n = 3$
0	-	-	0.278 37	1.006 51
1	0.693 67	1.415 53	1.891 17	2.246 93
2	1.963 08	2.348 29	-	-

对于 $\theta = 1$：

μ	$n = 0$	$n = 1$	$n = 2$	$n = 3$
0	-	-	0.236 38	0.970 14
1	0.711 68	1.420 12	1.889 97	2.242 98
2	1.967 12	2.350 22	-	-

在下表中，列出了 Chladni 发现的音调比例，并与计算结果进行了比较。它们是……

- Kirchhoff, on the equilibrium and motion of an elastic disk.

For $\theta = \frac{1}{2}$, i.e., $\gamma = \frac{4}{3}$ (according to Poisson):

n	0	1	2	3
$\log A_1$	0.664 2079	1.348 0266	0.265 0703	0.973 3073
$\log A_2$	2.535 1132	3.588 0591	2.061 5798	3.105 2465
$\log A_3$	5.097 3650	6.406 5347	4.588 9514	5.868 2055
$\log A_4$	8.149 924	9.655 978	7.620 8431	9.083 199
$\log A_5$	11.583 4	13.249 6	11.040 582	12.653 148
$\log A_6$	15.33	17.130	14.776 04	16.516 13
$\log A_7$	21.3	18.778 0	20.629 0	-
$\log A_8$	-	23.01	24.96	-
$\log A_9$	-	-	29.48	-

For $\theta = 1$, i.e., $\gamma = \frac{3}{2}$ (according to Wertheim):

n	0	1	2	3
$\log A_1$	0.681 2413	1.352 1826	0.224 2682	0.939 3022
$\log A_2$	2.556 3026	3.594 0911	2.018 9332	3.066 4444
$\log A_3$	5.120 4304	6.413 6351	4.546 2236	5.827 9058
$\log A_4$	8.174 058	9.663 768	7.578 3037	9.042 324
$\log A_5$	11.608 3	13.257 8	10.998 263	12.612 037
$\log A_6$	15.35	17.138	14.733 92	16.474 93
$\log A_7$	21.3	18.736	20.587 7	-
$\log A_8$	-	22.97	24.92	-
$\log A_9$	-	-	29.44	-

The following values of $\log(\lambda_{n\mu} l)^4$ are derived from this:

For $\theta = \frac{1}{2}$:

μ	$n = 0$	$n = 1$	$n = 2$	$n = 3$
0	-	-	0.278 37	1.006 51
1	0.693 67	1.415 53	1.891 17	2.246 93
2	1.963 08	2.348 29	-	-

For $\theta = 1$:

μ	$n = 0$	$n = 1$	$n = 2$	$n = 3$
0	-	-	0.236 38	0.970 14
1	0.711 68	1.420 12	1.889 97	2.242 98
2	1.967 12	2.350 22	-	-

In the following table, the tone ratios found by Chladni are listed together with those given by the calculation. They are...

给出一个最低音为 C 的盘子可以发出的音调。在标有 Ch. 的列中是 Chladni 观察到的音调，在标有 P. 的列中是基于假设 $\theta = \frac{1}{2}$ 计算出的音调，在标有 W. 的列中是基于假设 $\theta = 1$ 计算出的音调。所有数据都对应于平均温度 *)。每个计算出的音调通过其下方紧邻的音阶中的音调来表示，并附上 + 或 -，具体取决于它比该音调略高或略低。

μ	$n = 0$	$n = 1$	$n = 2$	$n = 3$
	Ch.	P.	W.	Ch.
0	—	—	—	—
1	Gis	Gis+	A+	b
2	gis+	b-	b+	e+

观察到的音调与两种计算结果之间存在显著差异。观察到的音调与基于 Poisson 假设 ($\theta = \frac{1}{2}$) 计算的结果更接近，但与基于 Wertheim 假设 ($\theta = 1$) 计算的结果相比，偏差较大，以至于不能从中得出反对这一假设的结论。

现在我将比较理论对节点线给出的一些数值结果与相应的观测结果。Srehlke 教授好心地提供了他进行的一些非常精确测量的结果，这些测量是在两个圆形玻璃盘上进行的。这些盘子制作得非常精细，就像他在 Dove's Repertorium 第三卷第 113 页中提到的方形盘一样；其中一个直径约为 6 英寸，厚度为 1 线，另一个直径为 7 英寸，厚度为 1.1 线。为了证明盘子的完美性和测量方法的准确性，可以通过测量不同直径的节点圆（不包括节点直径）得到以下数字的微小差异。

*) Chladni 在他的声学著作中没有明确说明他的数据来自哪个平均温度，但似乎毫无疑问，它们确实如此。

The tones that a disk can produce, whose lowest tone is C, are given. In the columns labeled Ch., the tones observed by Chladni are found; in those labeled P., the tones calculated under the assumption $\theta = \frac{1}{2}$; and in those labeled W., the tones calculated under the assumption $\theta = 1$. All data refer to the average temperature *). Each calculated tone is indicated by the nearest tone below it in the scale, with a + or - added according to whether it was slightly higher or lower than this tone.

μ	$n = 0$	$n = 1$	$n = 2$	$n = 3$
	Ch.	P.	W.	Ch.
0	—	—	—	—
1	Gis	Gis+	A+	b
2	gis+	b-	b+	e+

There are considerable deviations between the observed tones and those determined by the two calculations. The observed tones agree somewhat better with those calculated from Poisson's assumption ($\theta = \frac{1}{2}$) than with those calculated from Wertheim's assumption ($\theta = 1$); however, the deviation in the former case is too large to draw any conclusion against this assumption.

Now I turn to the comparison of some numerical results which the theory gives regarding the nodal lines with the corresponding results of observation. Professor Srehlke has kindly provided me with the results of some measurements of outstanding accuracy which he made on two circular glass disks. These disks were manufactured with the same care as the square disks on which he made the measurements mentioned in Dove's Repertorium Vol. III, p. 113; one had a diameter of about 6 inches and a thickness of 1 line, the other a diameter of 7 inches and a thickness of 1.1 lines. As proof of the perfection of the disks and the accuracy of the measurement method, the small differences in the following numbers, which were found by measuring different diameters of the nodal circles without nodal diameters on a disk, can serve.

*) Chladni does not explicitly state in his acoustics from which average temperature his data are taken; but it seems unquestionable that they do so.

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正面： 反面：
24^L, 415 24^L, 42
43 44
44 425
425 43
405 415
平均值 24^L, 423 24^L, 426

正如标记为I的这个盘子一样规律地表现，另一个7英寸的盘子（称为II）也表现出同样的规律性。我将把这两个盘子产生的节点圆半径的值与根据Poisson假设或Wertheim假设计算出的值一起列出。

观测	计算
I	II
n=0, μ=1	I.0,6792
n=1, μ=1	I.0,7811

对应于音调(n=0, μ=1)的节点圆半径也被Savart测量过；他在三个不同的盘子上得到了以下值：

l.0,6819, l.0,6798, l.0,6812.

这些数据在Poisson关于圆形板振动的研究中被提及。Poisson在那里假设θ=½的情况下，计算了n=0时最低音调及其对应的节点圆。

从Wertheim假设推导出的结果与从Poisson假设推导出的结果相差不大；与Strehlke的观测结果相比，它们的一致性更好。在我看来，这并不反对Poisson的假设，因为理论与实验之间的完全一致是无法期望的，因为用于实验的盘子并不完全具备理论中所假设的所有性质。

除了上述内容外，Strehlke先生还向我提供了其他一些不太完美的盘子的测量结果。我将列出这些结果，并附上相应的……

Front side: Back side:
24^L, 415 24^L, 42
43 44
44 425
425 43
405 415
Average 24^L, 423 24^L, 426

Just as regularly as this disk, which may be denoted by I, the other 7-inch disk, which shall be called II, also showed itself. I will list the values of the radii of the nodal circles produced by these two disks together with the values given by the calculation based on Poisson's or Wertheim's assumption of θ.

Observation	Calculation
I	II
n=0, μ=1	l.0,6792
n=1, μ=1	l.0,7811

The radius of the nodal circle corresponding to the tone (n=0, μ=1) was also measured by Savart; he found the following values for three different disks:

l.0,6819, l.0,6798, l.0,6812.

These data are mentioned by Poisson in his investigations about the vibrations of a circular plate in the above-cited treatise. Poisson calculated there the lowest tones and the corresponding nodal circles for the case n=0 under the assumption θ=½.

The results derived from the Wertheim assumption deviate only slightly from those derived from the Poisson assumption; they agree even better with Strehlke's observations than these do. In my opinion, this does not speak against the Poisson assumption, because a complete agreement between theory and experiment cannot be expected, since the disks subjected to the experiment do not possess all the properties assumed in the theory in strictness.

In addition to the above, Mr. Strehlke has informed me of the results of some other measurements made on less perfect disks. I will list these results along with the corresponding...

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以下是计算在假设 $\theta = \frac{1}{2}$ 和 $\theta = 1$ 下给出的数值。

节点圆的半径。

n, μ	**观测值**	**计算值**
		$\theta = \frac{1}{2}$
$n = 1, \mu = 1$	0.781	0.783
$n = 2, \mu = 1$	0.79	0.81
$n = 3, \mu = 1$	0.838	0.842
$n = 1, \mu = 2$	0.488	0.492
	0.869	0.869

在此处设定盘子的半径为 1。

日期：1850年1月。

The following are the numbers given by the calculation under the assumption $\theta = \frac{1}{2}$ and $\theta = 1$.

Radii of the nodal circles.

n, μ	**Observation**	**Calculation**
		$\theta = \frac{1}{2}$
$n = 1, \mu = 1$	0.781	0.783
$n = 2, \mu = 1$	0.79	0.81
$n = 3, \mu = 1$	0.838	0.842
$n = 1, \mu = 2$	0.488	0.492
	0.869	0.869

Here the radius of the disk is set to 1.

Date: January 1850.